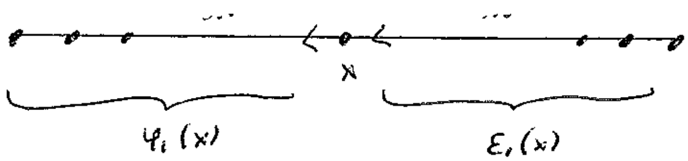
Lecture 17 Friday Oct 11

$\Phi = A_2$ $GL(3)$ $\mathcal{B}_\lambda$ $\lambda = (7, 3)$

For $x \in \mathcal{B}_\lambda$ recall

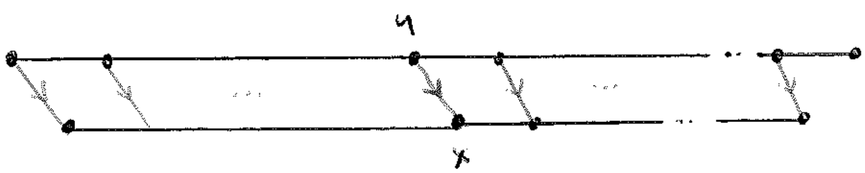


For

compare

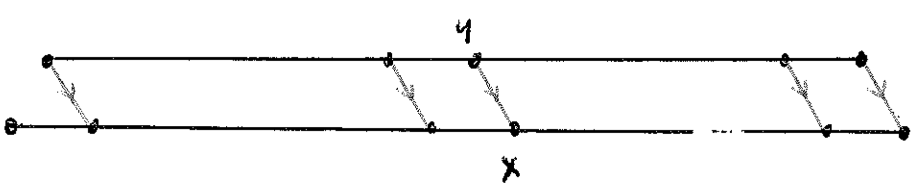
$$\varphi_1(x),\ \varepsilon_1(x),\ \varphi_1(y),\ \varepsilon_1(y)$$

<u>Case</u> x has level ≤ 0.



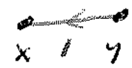
$$\varphi_1(x) = \varphi_1(y), \qquad \varepsilon_1(x) = \varepsilon_1(y) - 1$$

<u>Case</u> x has level > 0

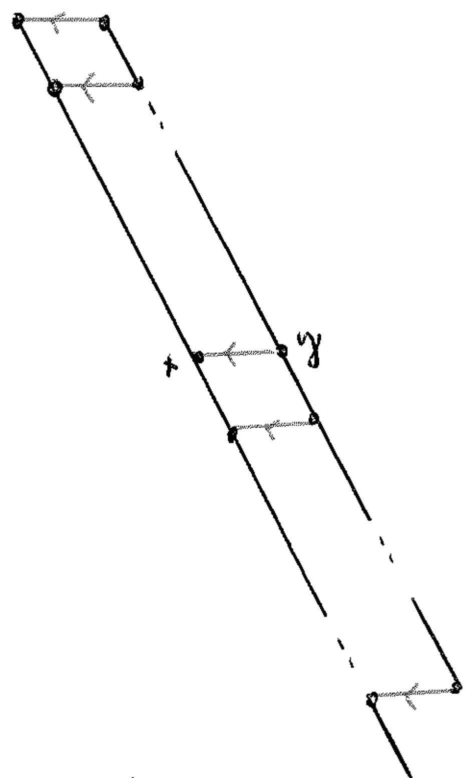


$$\varphi_1(x) = \varphi_1(y) + 1, \qquad \varepsilon_1(x) = \varepsilon_1(y)$$

Similarly, for

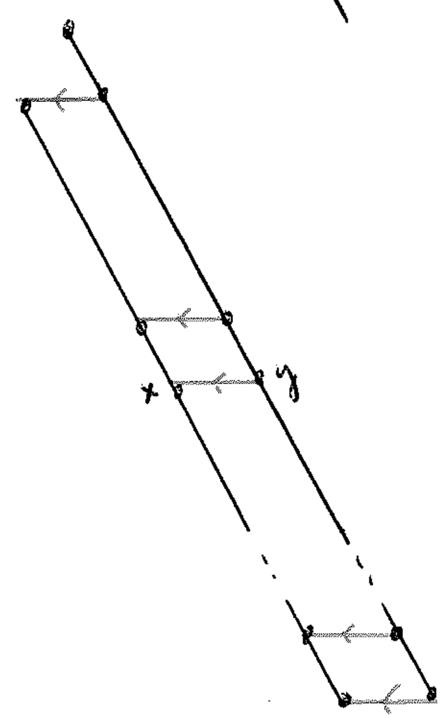


<u>Case</u> x has level < 0



$$\varphi_2(y) = \varphi_2(x) - 1$$
$$\varepsilon_2(y) = \varepsilon_2(x)$$

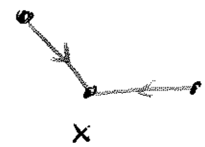
<u>Case</u> x has level ≥ 0

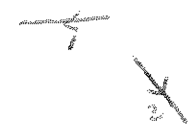


$$\varphi_2(y) = \varphi_2(x)$$
$$\varepsilon_2(y) = \varepsilon_2(x) + 1$$

$\Phi = A_2$, $\quad GL(3)$, $\quad \mathcal{B}_\lambda$, $\quad \lambda = (7,3)$

Assume





level of x

	<0	$=0$	>0
$\varphi_1(e_2x) - \varphi_1(x)$	0	0	-1
$\varepsilon_1(e_2x) - \varepsilon_1(x)$	1	1	0
$\varphi_2(e_1x) - \varphi_2(x)$	-1	0	0
$\varepsilon_2(e_1x) - \varepsilon_2(x)$	0	1	1

$\Phi = A_2, \qquad GL(3), \qquad \mathcal{B}_\lambda, \qquad \lambda = (7,3)$

Assume

Then

(i) x has level < 0 iff

$$\varepsilon_1(e_2 x) = \varepsilon_1(x) + 1 \quad \text{and} \quad \varepsilon_2(e_1 x) = \varepsilon_2(x)$$

(ii) x has level 0 iff

$$\varepsilon_1(e_2 x) = \varepsilon_1(x) + 1 \quad \text{and} \quad \varepsilon_2(e_1 x) = \varepsilon_2(x) + 1$$

(iii) x has level > 0 iff

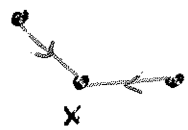
$$\varepsilon_1(e_2 x) = \varepsilon_1(x) \quad \text{and} \quad \varepsilon_2(e_1 x) = \varepsilon_2(x) + 1$$

By (i)–(iii) we can determine the level of x from

$$\varepsilon_1(x), \quad \varepsilon_2(x), \quad \varepsilon_1(e_2 x), \quad \varepsilon_2(e_1 x)$$

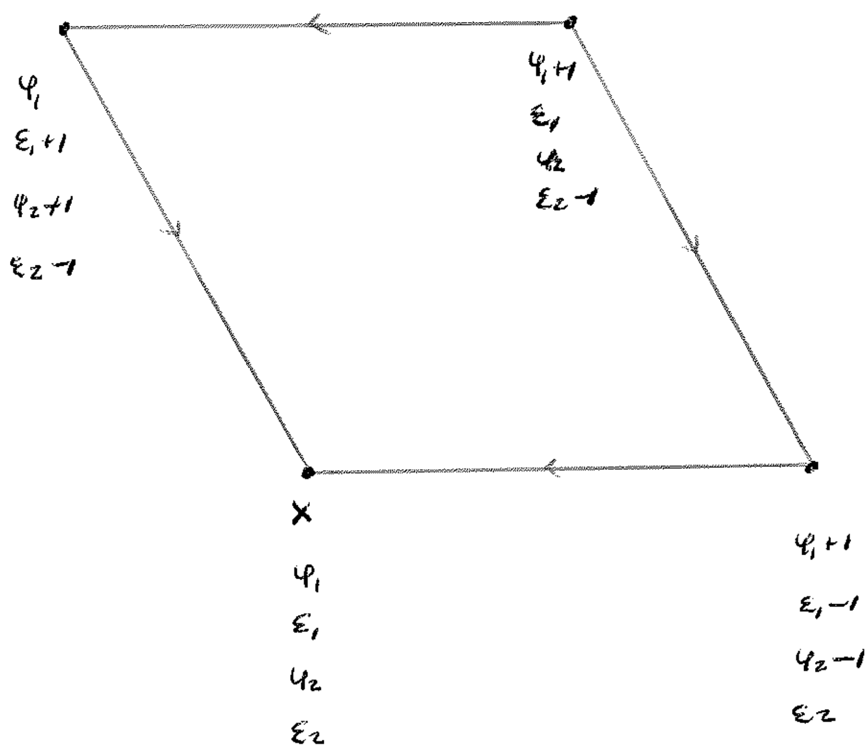
$\Phi = A_2$, $GL(3)$, B_λ, $\lambda = (7, 3)$

Assume



$\varepsilon_1(e_2 x) = \varepsilon_1(x) + 1$,
$\varepsilon_2(e_1 x) = \varepsilon_2(x)$

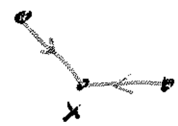
Then



Here x has level < 0

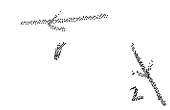
$\Phi = A_2$, $GL(3)$, B_λ $\lambda = (7,3)$

Assume

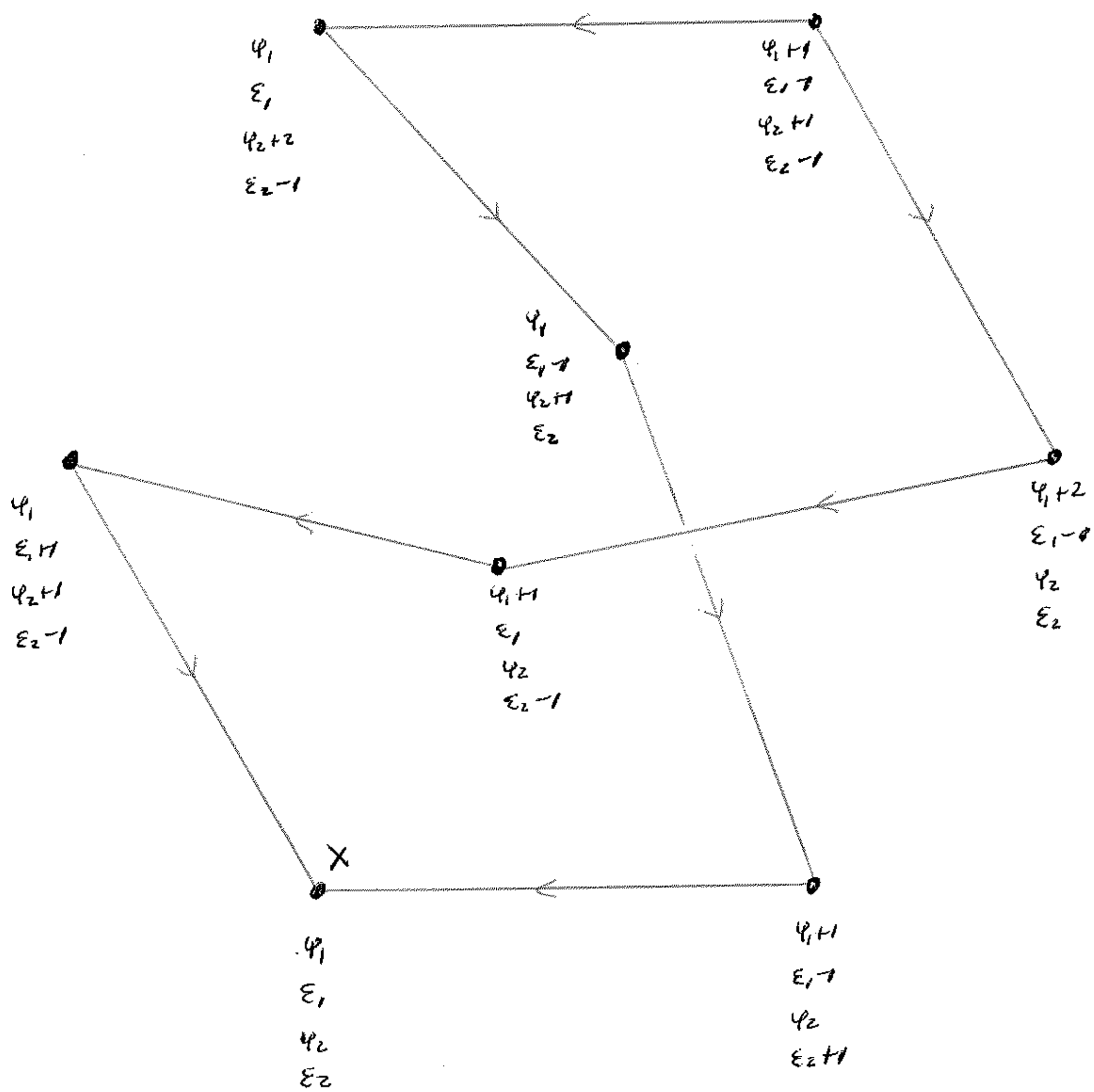


$\varepsilon_1(e_2 x) = \varepsilon_1(x) + 1$

$\varepsilon_2(e_1 x) = \varepsilon_2(x) + 1$



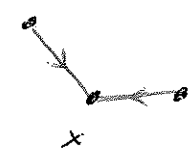
Then



Here x has level 0

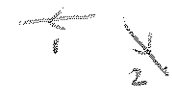
$\Phi = A_2$, $\quad GL(3)$, $\quad B_\lambda$, $\quad \lambda = (7, 3)$

Assume

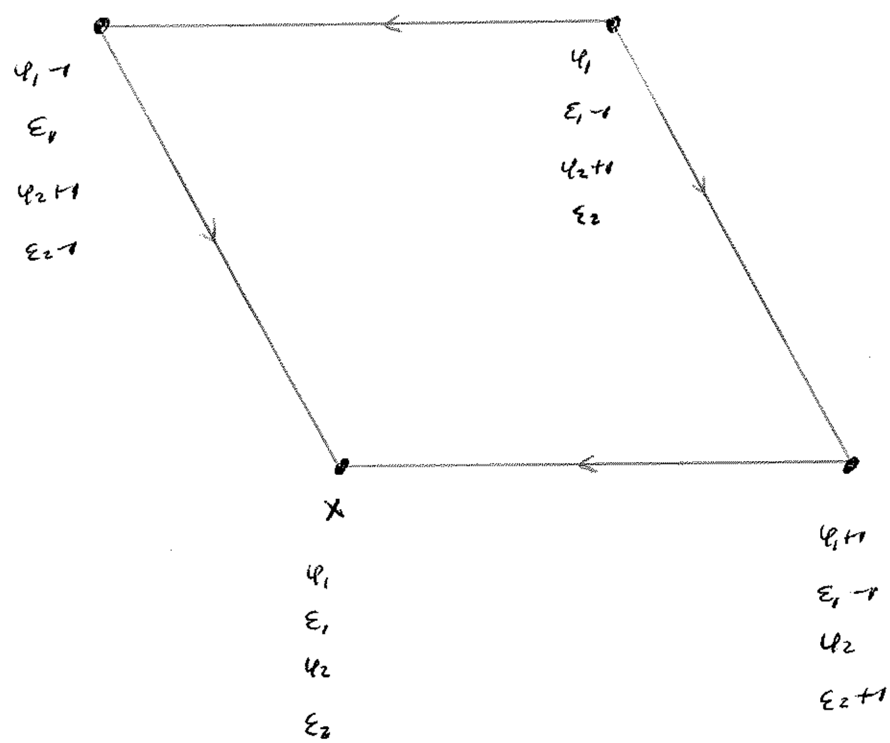


$\varepsilon_1(e_2 x) = \varepsilon_1(x)$

$\varepsilon_2(e_1 x) = \varepsilon_2(x) + 1$



then



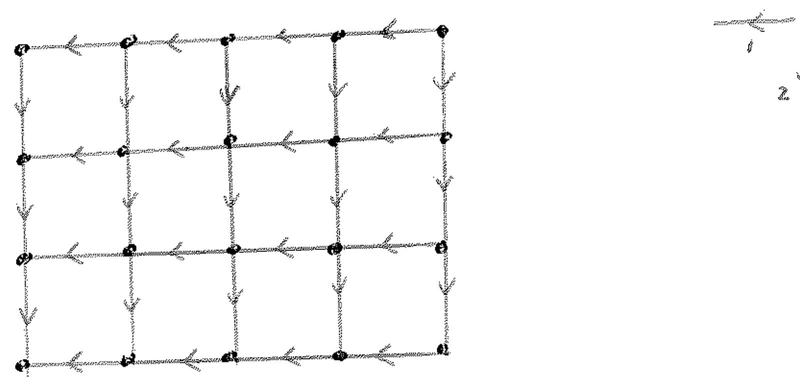
Here x has level > 0

We are done with the B_λ example.

Next we describe a Stembridge crystal B

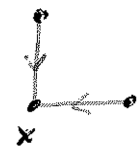
for $\Phi = A_1 \times A_1$ so $\Phi = \{ \pm\alpha_1, \pm\alpha_2 \mid \langle \alpha_1, \alpha_2 \rangle = 0 \}$

B is semi normal with crystal graph a disjoint union of rectangles:

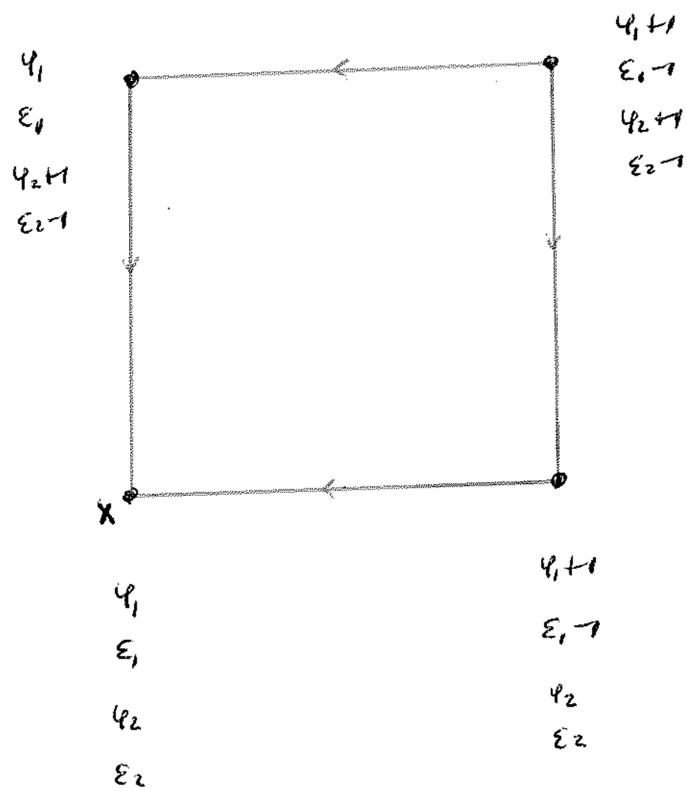


For the above example B,

assume



then



We now describe the Stembridge axioms.

Until further notice root system Φ is <u>simply laced</u>

DEF A crystal $\mathcal{B}$ for Φ is <u>Stembridge</u> whenever $\mathcal{B}$ is seminormal and satisfies S1 – S3, S1' – S3' below

S1 For $x, y \in \mathcal{B}$ and $i \in I$ st

$$x \xleftarrow{i} y$$

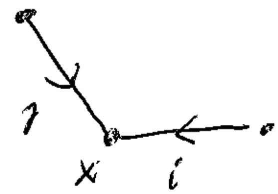
then for $j \in I \setminus \{i\}$

$$\varepsilon_j(y) = \varepsilon_j(x) \qquad *$$

$$\text{or} \quad \varepsilon_j(y) = \varepsilon_j(x) + 1 \qquad **$$

If ** occurs then $\langle \alpha_i, \alpha_j \rangle \neq 0$

S2 For $x \in B$ and distinct $i, j \in I$ s.t

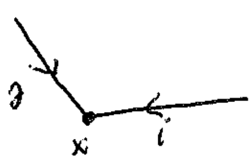


and $\varepsilon_j(e_i x) = \varepsilon_j(x)$

then

$$e_i e_j x = e_j e_i x \quad \text{and} \quad \varphi_i(e_j x) = \varphi_i(x)$$

S3 For $x \in B$ and distinct $i, j \in I$ s.t



and $\varepsilon_j(e_i x) = \varepsilon_j(x) + 1,$
$\varepsilon_i(e_j x) = \varepsilon_i(x) + 1$

then

$$e_i e_j^2 e_i x = e_j e_i^2 e_j x \neq \phi$$

and

$$\varphi_i(e_j x) = \varphi_i(e_j^2 e_i x),$$
$$\varphi_j(e_i x) = \varphi_j(e_i^2 e_j x)$$

S1' For $x, y \in B$ and $i \in I$ st

$$x \xleftarrow{i} y$$

Then for $j \in I \setminus \{i\}$

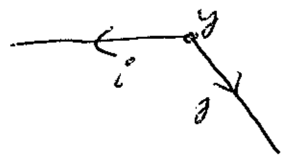
$$\varphi_j(x) = \varphi_j(y) \qquad *$$

or

$$\varphi_j(x) = \varphi_j(y) + 1 \qquad **$$

If ** occurs then $\langle \alpha_i, \alpha_j \rangle \neq 0$

$S2'$ For $y \in B$ and dist $i, j \in I$ st

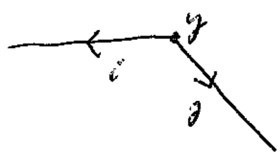


$$\varphi_j(f_i y) = \varphi_j(y)$$

Then

$$f_i f_j y = f_j f_i y \quad \text{and} \quad \varepsilon_i(f_j y) = \varepsilon_i(y)$$

$S3'$ For $y \in B$ and dist $i, j \in I$ st



and

$$\varphi_j(f_i y) = \varphi_j(y) + 1$$

$$\varphi_i(f_j y) = \varphi_i(y) + 1$$

Then

$$f_i f_j^2 f_i y = f_j f_i^2 f_j y \neq \phi$$

and

$$\varepsilon_i(f_j y) = \varepsilon_i(f_j^2 f_i y),$$

$$\varepsilon_j(f_i y) = \varepsilon_j(f_i^2 f_j y)$$

We just defined a Stembridge crystal.

To get a weak Stembridge crystal, we replace the seminormal assumption by:

S0: For $x \in \mathcal{B}$ and $i \in I$,
$e_i(x) = \phi$ implies $\varepsilon_i(x) = 0$

S0': For $x \in \mathcal{B}$ and $i \in I$,
$f_i(x) = \phi$ implies $\varphi_i(x) = 0$.

LEM Given a Stembridge crystal B for a simply connected root system then each full subcrystal of B (ie disjoint union of connected components) is Stembridge.

Pf By constr.

Ex For $\Phi = A_r$ or D_r the standard crystal is Stembridge.

pf Routine.

Thm. Given Stembridge crystals B, C for a simply laced root systems then the crystal $B \otimes C$ is Stembridge

pf very tedious, see txt

LEM For $\Phi = A_{r}$, $GL(r+1)$ $n = r+1$

Then for each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda$ the crystal $\mathcal{B}_\lambda$ is Stembridge.

pf $\mathcal{B}_\lambda$ iso connected component of $\mathcal{B}^{\otimes|\lambda|}$ where $\mathcal{B}$ is standard crystal for Φ

$\mathcal{B}$ is Stembridge

$\mathcal{B}^{\otimes|\lambda|}$ is Stembridge

$\mathcal{B}_\lambda$ is Stembridge $\square$