Lecture 16 Wednesday Oct 9

Given $r, k \geq 1$

Consider sequence

$$x = a_1 a_2 \cdots a_k \qquad a_i \in \{1, 2, \ldots, n\} \qquad n = r+1 \qquad *$$

View

$$x \in B^{\otimes k} \qquad B = B_{(1)} \quad \text{for } A_r$$

x is contained in a unique connected component C of crystal $B^{\otimes k}$

Crystal C is iso to crystal B_λ for some partition $\lambda \vdash k$ with at most n parts

Call iso P

Crystal C has unique hw element $\bar{x}$; corresp standard tableau by $Q(\bar{x})$ has shp λ

The map

$$x \longrightarrow P(x),\ Q(\bar{x})$$

gives a bijection

$$B^{\otimes k} \longrightarrow \left\{ (P, Q) \;\middle|\; \begin{array}{l} P = \text{s st. tabl, entries in } 1,2,\dots,n \\ Q = \text{st tabl} \\ P, Q \text{ same shp } \lambda \vdash k \end{array} \right\}$$

As we will see, above bijection is exactly the

R S K corresp

Robinson Schensted Knuth

Ref to $*$, Consider special case in which $k = n$ and

$a_1, a_2, \dots, a_n$ mut dist

So $x = a_1 a_2 \cdots a_n$ is perm of $\{1, 2, \dots, n\}$

View $x \in S_n$.

Turns out s st tabl $P(x)$ is actually standard.

The above map

$$x \longrightarrow P(x),\; Q(\bar{x})$$

Gives bij

$$S_n \longrightarrow \left\{ (P, Q) \;\middle|\; \begin{array}{l} P, Q \text{ st tabl} \\ \text{same shp } \lambda \vdash n \end{array} \right\}$$

As we will see, this bij is exactly the

Robinson - Schensted corresp.

Ch 4: <u>Stembridge Crystals</u>

Given root data Φ, Λ, $\Sigma = \{\alpha_i\}_{i \in I}$

Recall partial order $\preceq$ on Λ:

$$\mu \preceq \lambda \quad \text{iff} \quad \lambda - \mu = \sum_{i \in I} a_i \alpha_i \qquad a_i \geq 0$$

Given crystal $\mathcal{B}$ for above root data

Define partial order $\preceq$ on $\mathcal{B}$:

For $x, y \in \mathcal{B}$

y covers x wrt $\preceq$ whenever $x \xleftarrow{i} y$

for some $i \in I$

So for $x, y \in \mathcal{B}$

$x \preceq y$ whenever $\exists$ directed path in crystal graph

$$x \xleftarrow{i_r} y_{r-1} \leftarrow \cdots \leftarrow y_2 \xleftarrow{i_2} y_1 \xleftarrow{i_1} y$$

$r \geq 0$, $\quad i_1, i_2, \ldots, i_r \in I$

In this case

$$wt(y) - wt(x) = \sum_{l=1}^{r} \alpha_{i_l}$$

So $\quad wt(x) \preceq wt(y)$

<u>Note</u> Resulting poset $\mathcal{B}$ is not a lattice in general.

Poset B is graded in following sense.

Recall Weyl vector $\rho \in \Lambda$

$$\langle \rho, \alpha_i^\vee \rangle = 1 \qquad i \in I$$

Weyl vector $\rho^\vee$ for $\Phi^\vee$ satisfies

$$\langle \rho^\vee, \alpha_i \rangle = 1 \qquad i \in I$$

Given $x, y \in B$ st y covers x in poset B:

$i \in I$

$$x \xleftarrow{\;i\;} y$$

so $\quad wt(y) - wt(x) = \alpha_i$

so $\quad \langle \rho^\vee, wt(y) \rangle - \langle \rho^\vee, wt(x) \rangle = \langle \rho^\vee, \alpha_i \rangle$
$= 1$

Next general goal

Recall for $\Phi = A_r$, type $GL(r+1)$

We defined some crystals $\mathcal{B}_\lambda$

We wish to characterize these $\mathcal{B}_\lambda$ axiomatically

without reference to λ

The axioms are due to Stembridge.

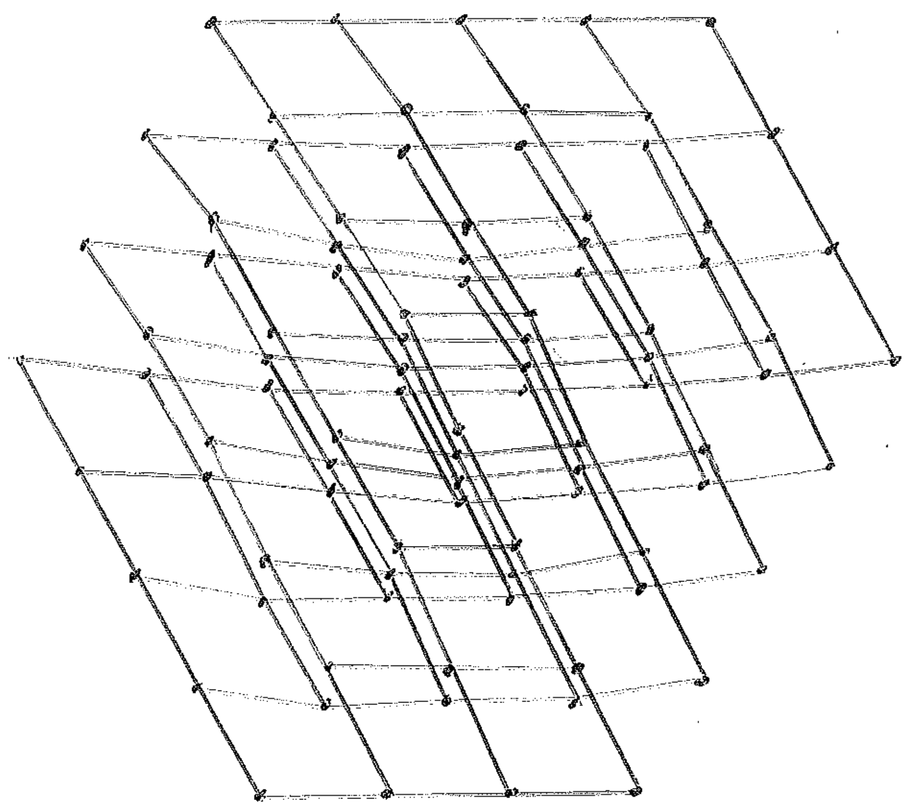
A crystal that satisfies the axioms is called Stembridge.

To motivate the axioms, we consider an example

$r=2$, $\lambda = (7,3)$

$\Phi = A_2 \qquad GL(3) \qquad B_\lambda \qquad \lambda = (7, 3)$

Next goal: investigate crystal graph B_λ



$\Phi = A_2, \quad GL(3), \quad B_\lambda, \quad \lambda = (7,3)$

Showing $\langle \cdot , \alpha_1^\vee \rangle$

$$\begin{array}{ccccccccccccccc}
 & & & -4 & & -2 & & 0 & & 2 & & 4 & & & \\
 & & -5 & & -3 & & -1 & & 1 & & 3 & & 5 & & \\
 & -6 & & -4 & & -2 & & 0 & & 2 & & 4 & & 6 & \\
-7 & & -5 & & -3 & & -1 & & 1 & & 3 & & 5 & & 7 \\
 & -6 & & -4 & & -2 & & 0 & & 2 & & 4 & & 6 & \\
 & & -5 & & -3 & & -1 & & 1 & & 3 & & 5 & & \\
 & & & -4 & & -2 & & 0 & & 2 & & 4 & & & \\
 & & & & -3 & & -1 & & 1 & & 3 & & & &
\end{array}$$

$\Phi = A_2$, $GL(3)$ B_λ, $\lambda = (7,3)$

Showing $\langle \cdot , \alpha_2^\vee \rangle$

7 6 5 4 3

6 5 4 3 2 1

5 4 3 2 1 0 -1

4 3 2 1 0 -1 -2 -3

2 1 0 -1 -2 -3 -4

0 -1 -2 -3 -4 -5

-2 -3 -4 -5 -6

-4 -5 -6 -7

$\Phi = A_2$, $GL(3)$, $\mathcal{B}_\lambda$ $\lambda = (7, 3)$

Showing number of vertices in $\mathcal{B}_\lambda$ with given wt

1 1 1 1 1

1 2 2 2 2 1

1 2 3 3 3 2 1

1 2 3 4 4 3 2 1

1 2 3 4 3 2 1

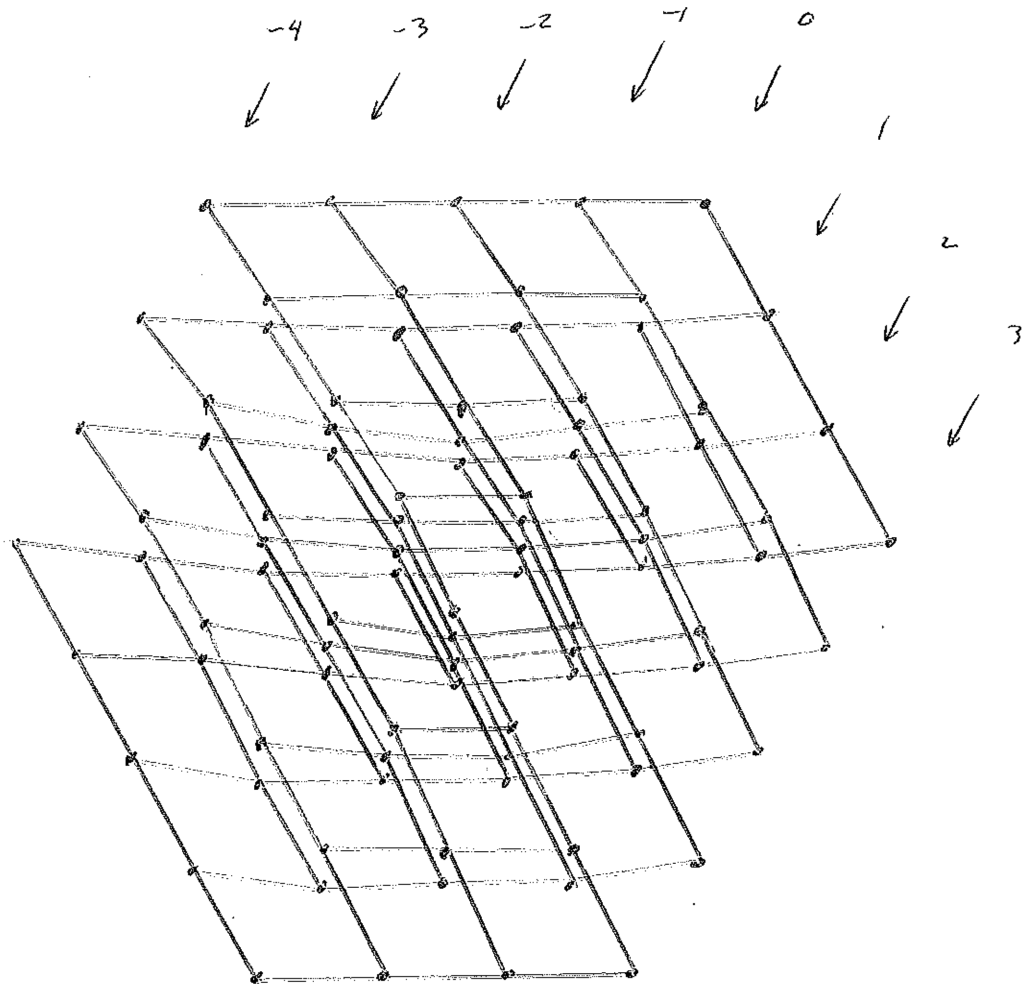
1 2 3 3 2 1

1 2 2 2 1

1 1 1 1

$\Phi = A_2 \qquad GL(3) \qquad B_\lambda \qquad \lambda = (7, 3)$

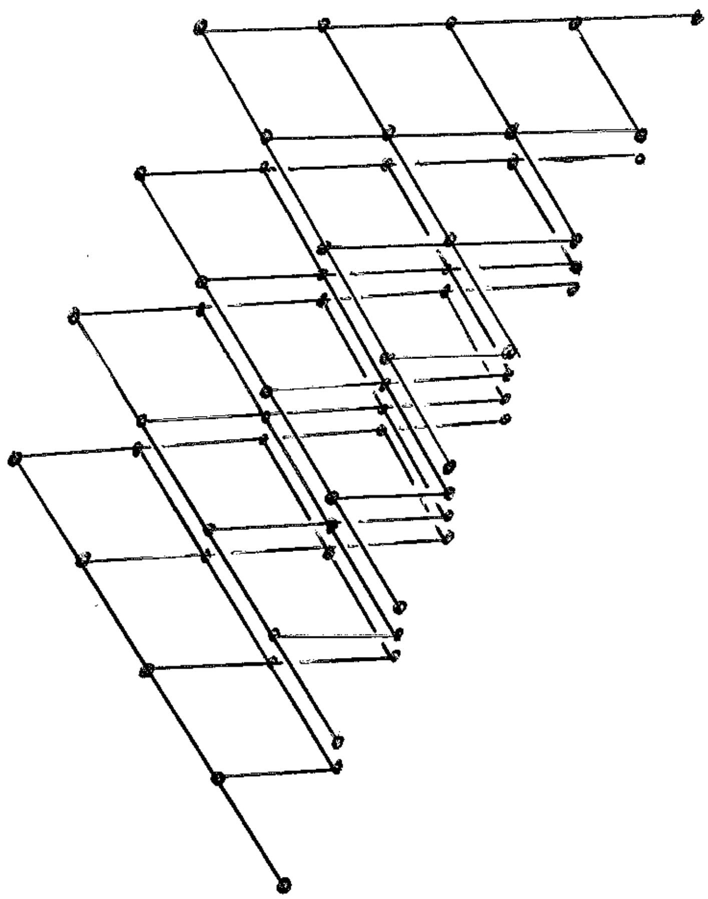
Define Levels



$$\text{level}(x) = \frac{\langle wt(x) - \lambda,\ \alpha_1^\vee - \alpha_2^\vee \rangle}{3} \qquad x \in B_\lambda$$

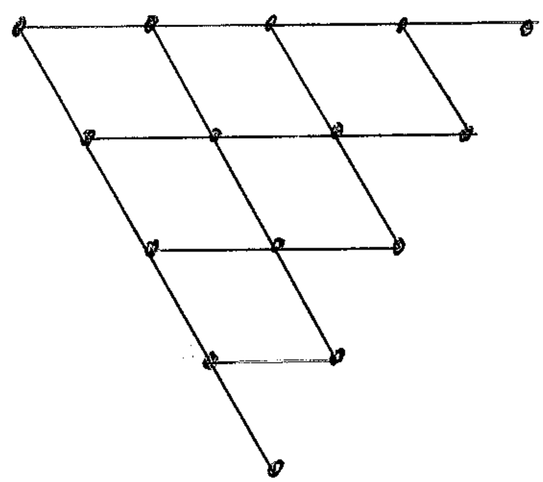
$\Phi = A_2 \qquad GL(3) \qquad B_\lambda \qquad \lambda = (7,3)$

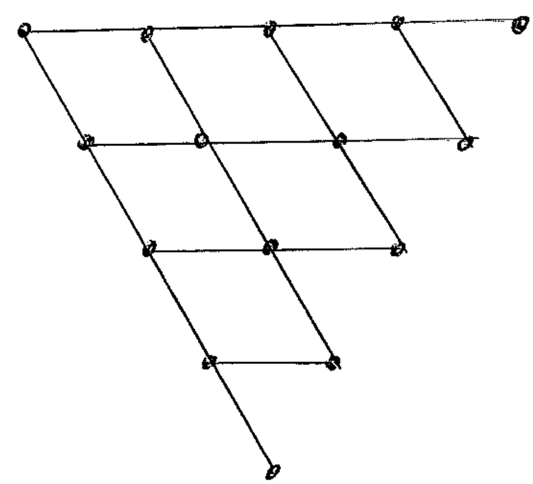
" subcrystal " induced by level ≤ 0

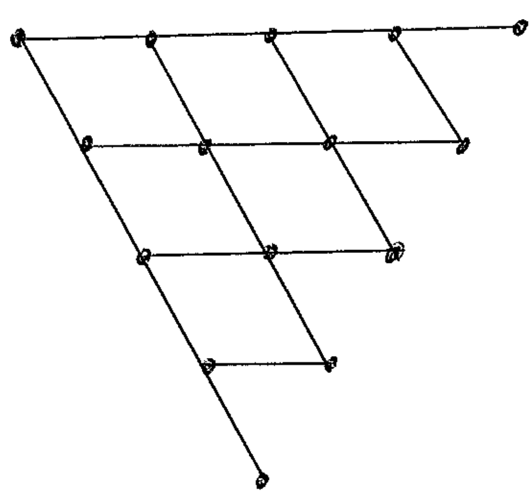


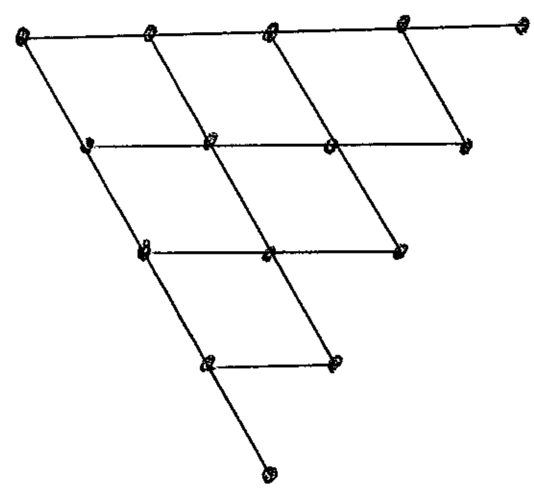
$\Phi = A_2$ $\quad GL(3) \quad B_\lambda \quad \lambda = (7, 3)$

subcrystal induced by level ≤ 0 is disjoint union of 4 connected components



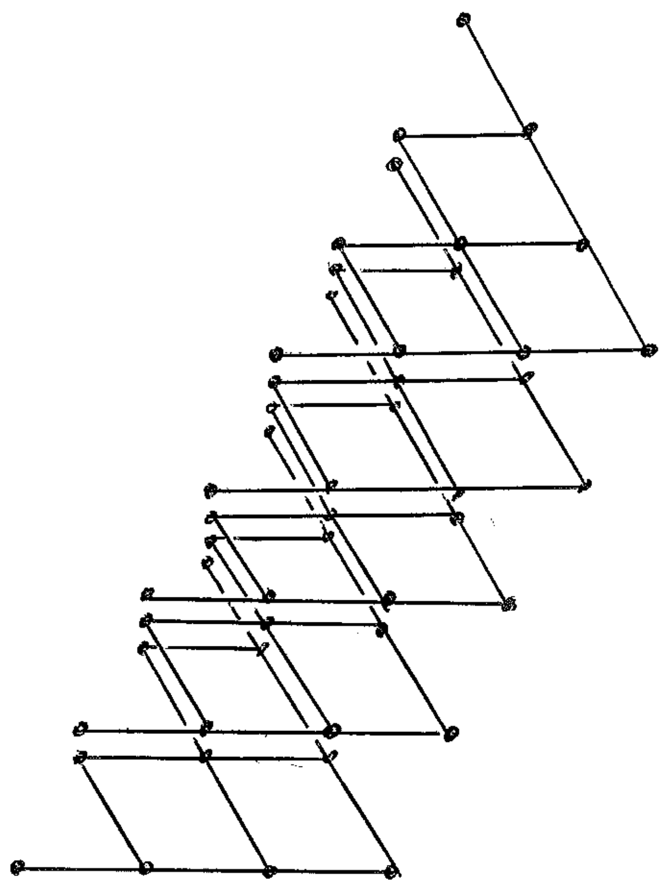






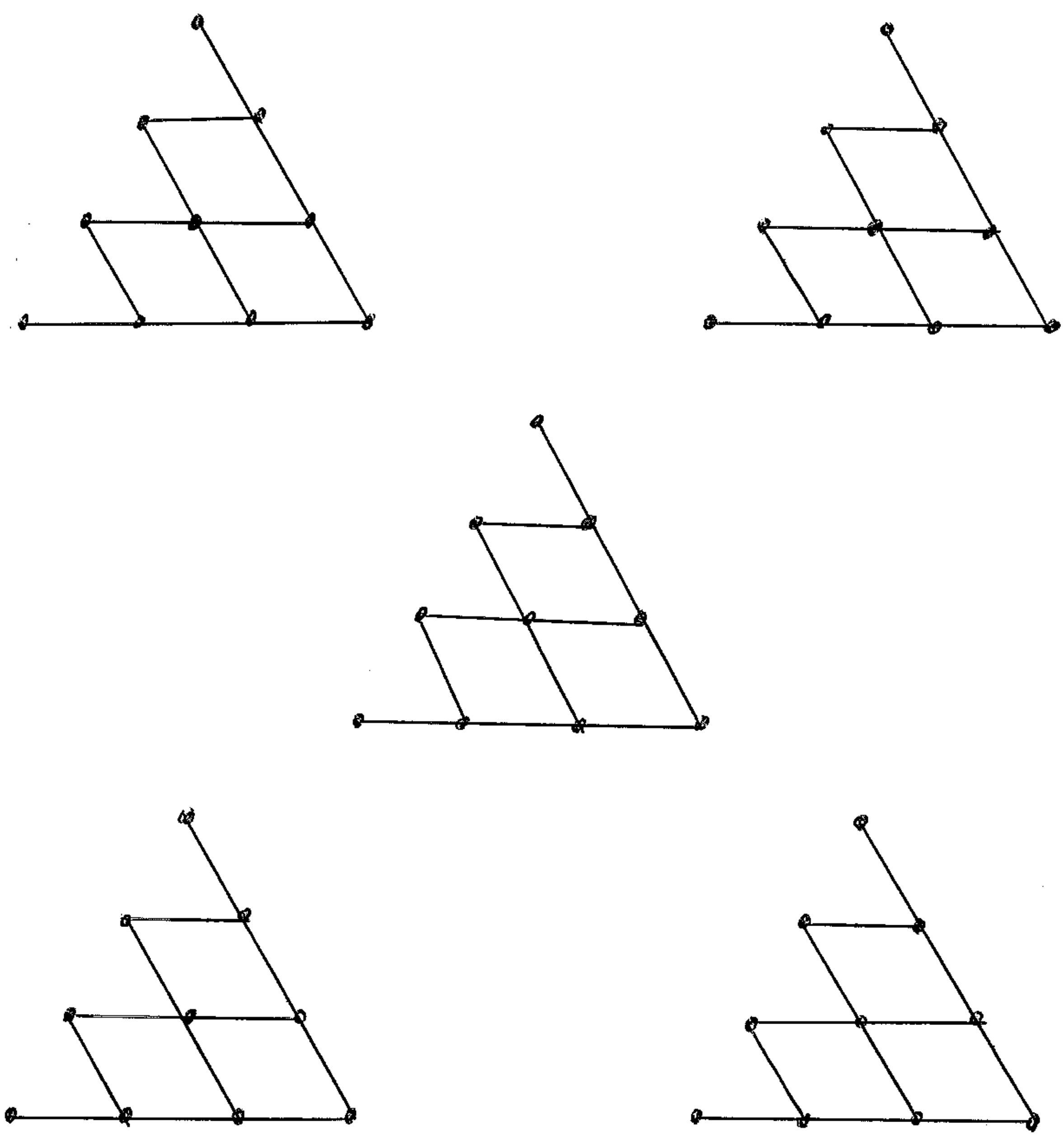
$\Phi = A_2$ $\quad GL(3) \quad B_\lambda \quad \lambda = (7, 3)$ 10/9/19

"subcrystal" induced on level ≥ 0



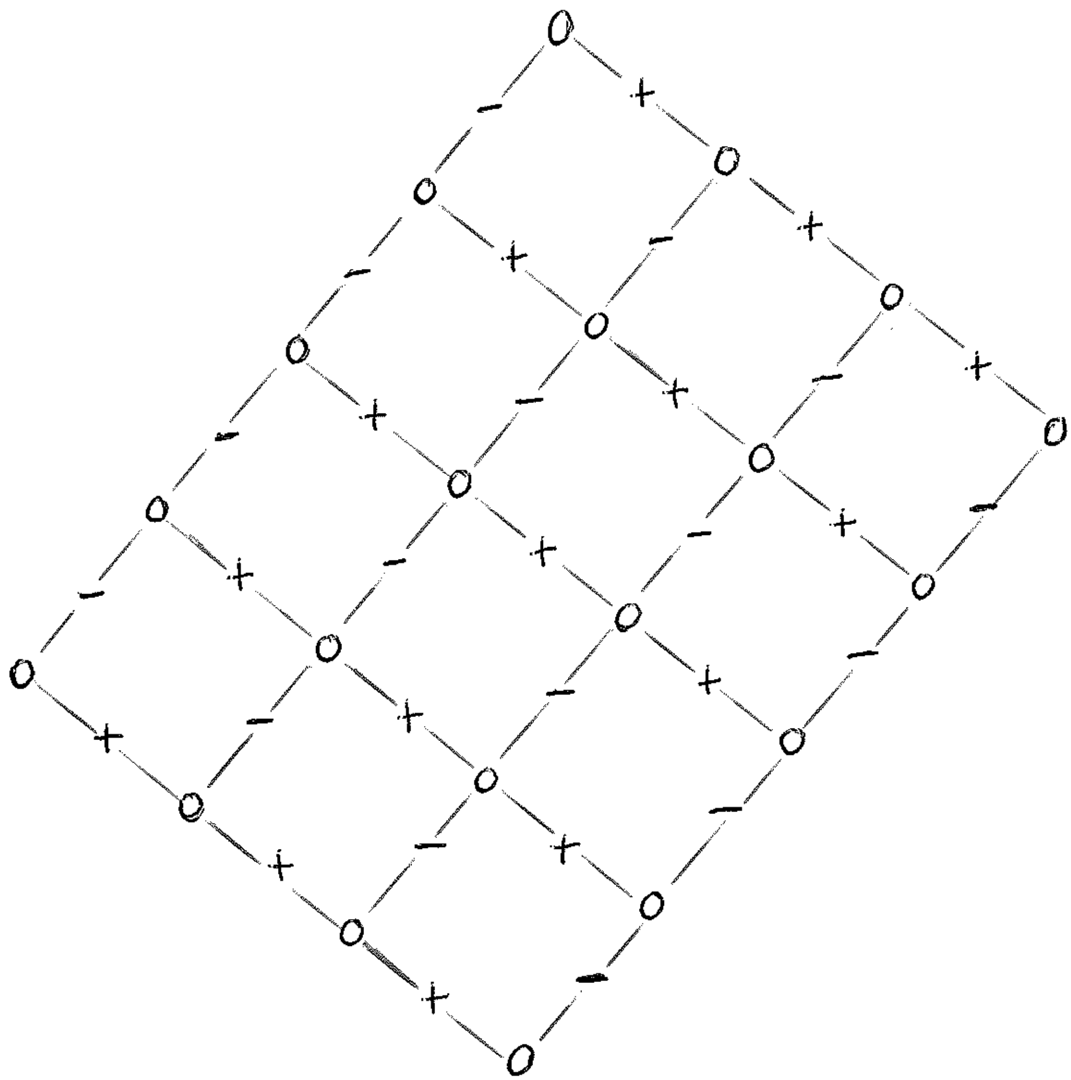
$\Phi = A_2$ $\quad GL(3) \quad \mathcal{B}_\lambda \quad \lambda = (7,3)$

Subcrystal induced by levels ≥ 0 is disjoint union of 5 connected components



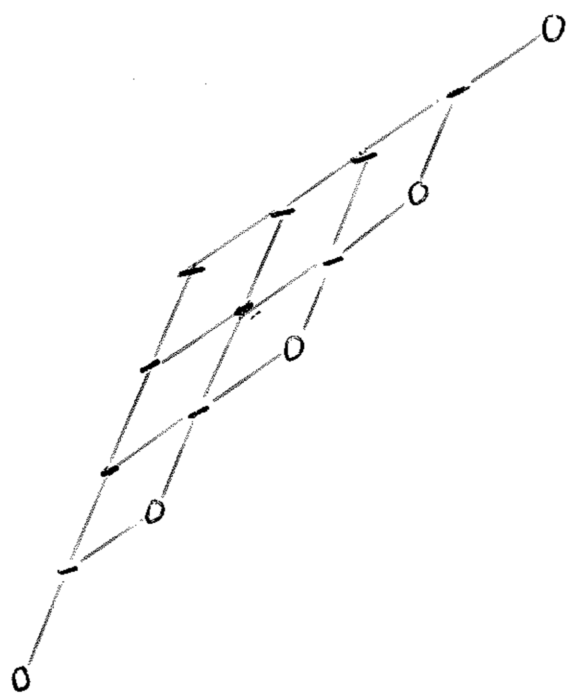
$\Phi = A_2 \qquad GL(3) \qquad B_\lambda \qquad \lambda = (7,3)$

Subcrystal induced on Levels $-1, 0, 1$



Key				
Level		-1	0	1
notation		$-$	0	$+$

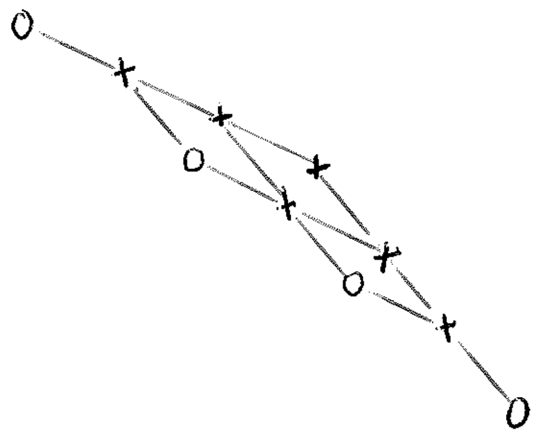
$\Phi = A_2$, $GL(3)$ $\mathcal{B}_\lambda$ $\lambda = (7,3)$



Ref to the subcrystal induced on levels $-1, 0, 1$

one of the above is attached to each NE-SW diagonal

$\Phi = A_2, \quad GL(3), \quad \mathcal{B}_\lambda, \quad \lambda = (7, 3)$



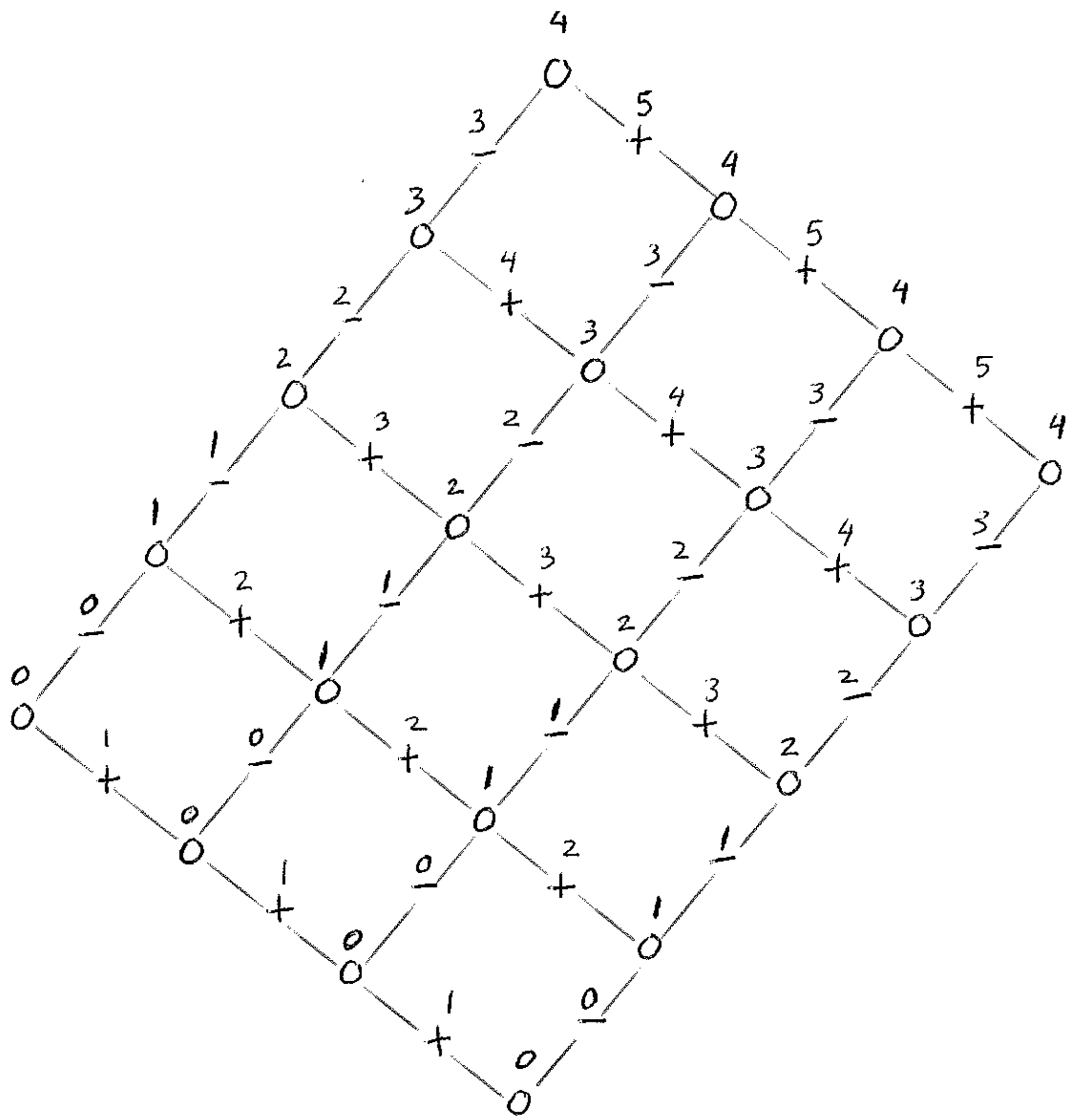
Ref to the subcrystal induced on Levels $-1, 0, 1$

one of the above is attached to each NW–SE diagonal

$\Phi = A_2$, $GL(3)$, $\mathcal{B}_\lambda$, $\lambda = (7,3)$

Subcrystal induced on levels $-1, 0, 1$

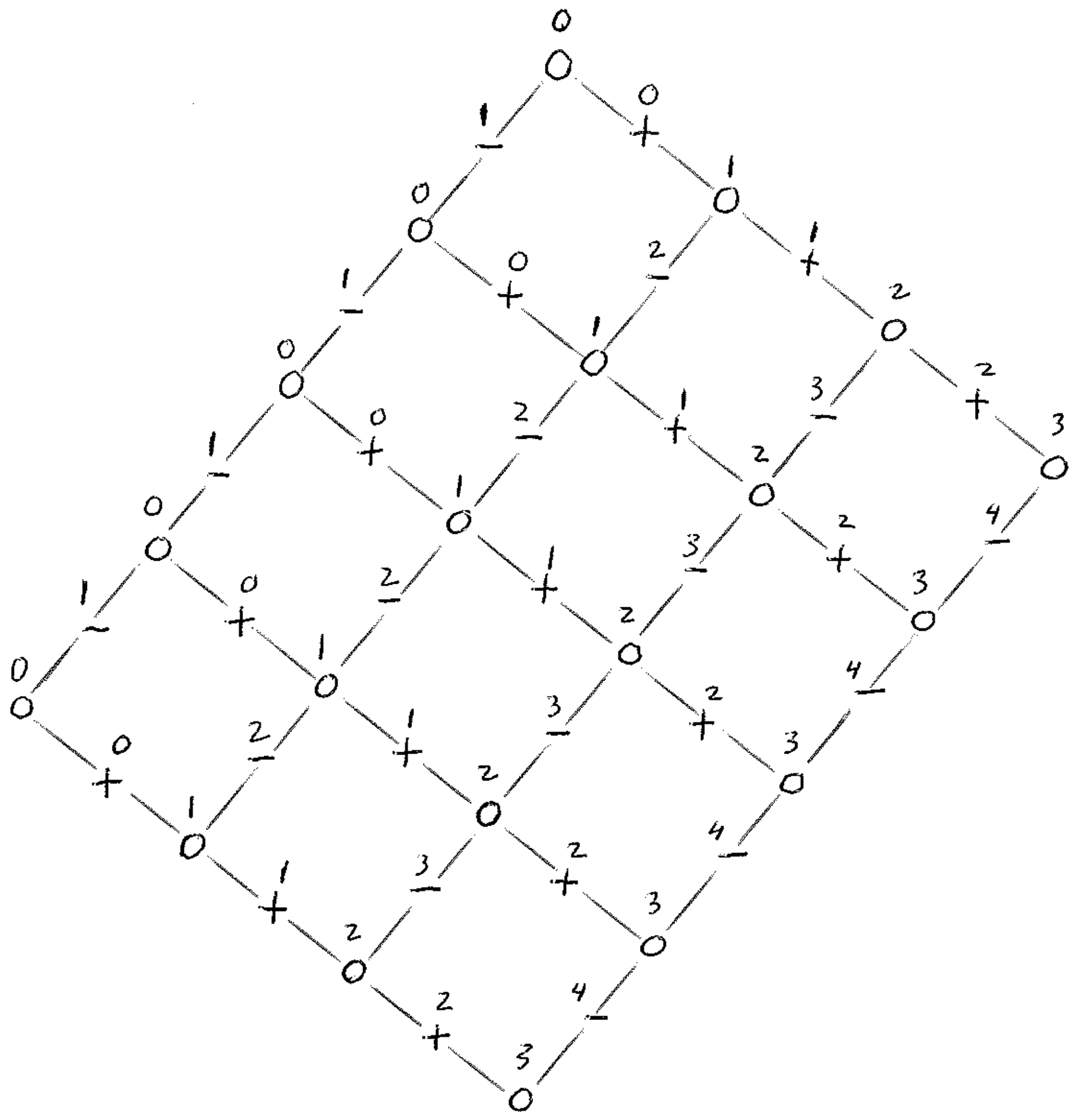
showing φ_1



$\Phi = A_2$, $GL(3)$, $\mathcal{B}_\lambda$ $\lambda = (7,3)$

Subcrystal induced on levels $-1, 0, 1$

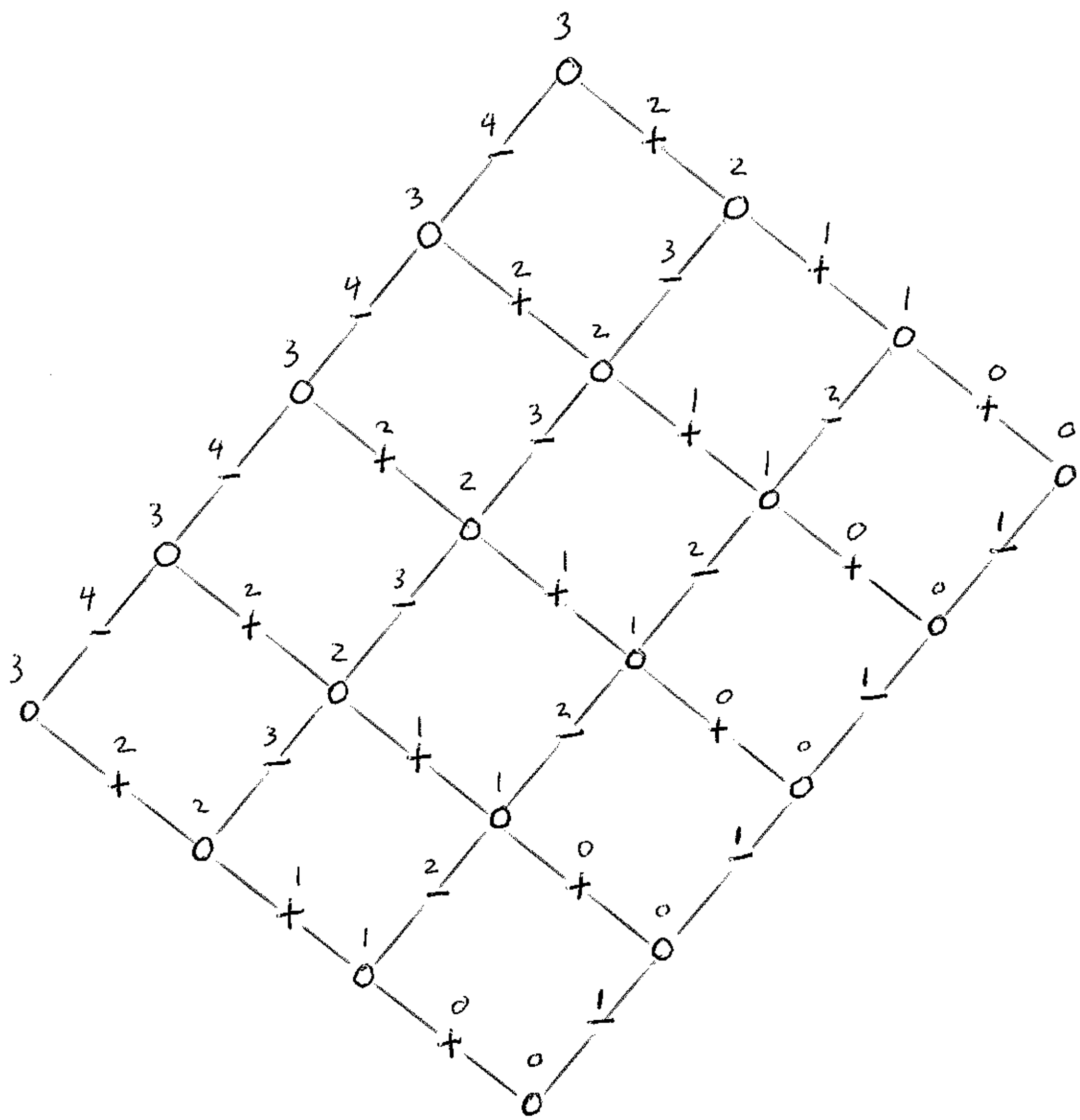
showing ε_1



$\Phi = A_2$, $\quad GL(3)$, $\quad \mathcal{B}_\lambda$, $\quad \lambda = (7,3)$

Subcrystal induced on levels $-1, 0, 1$

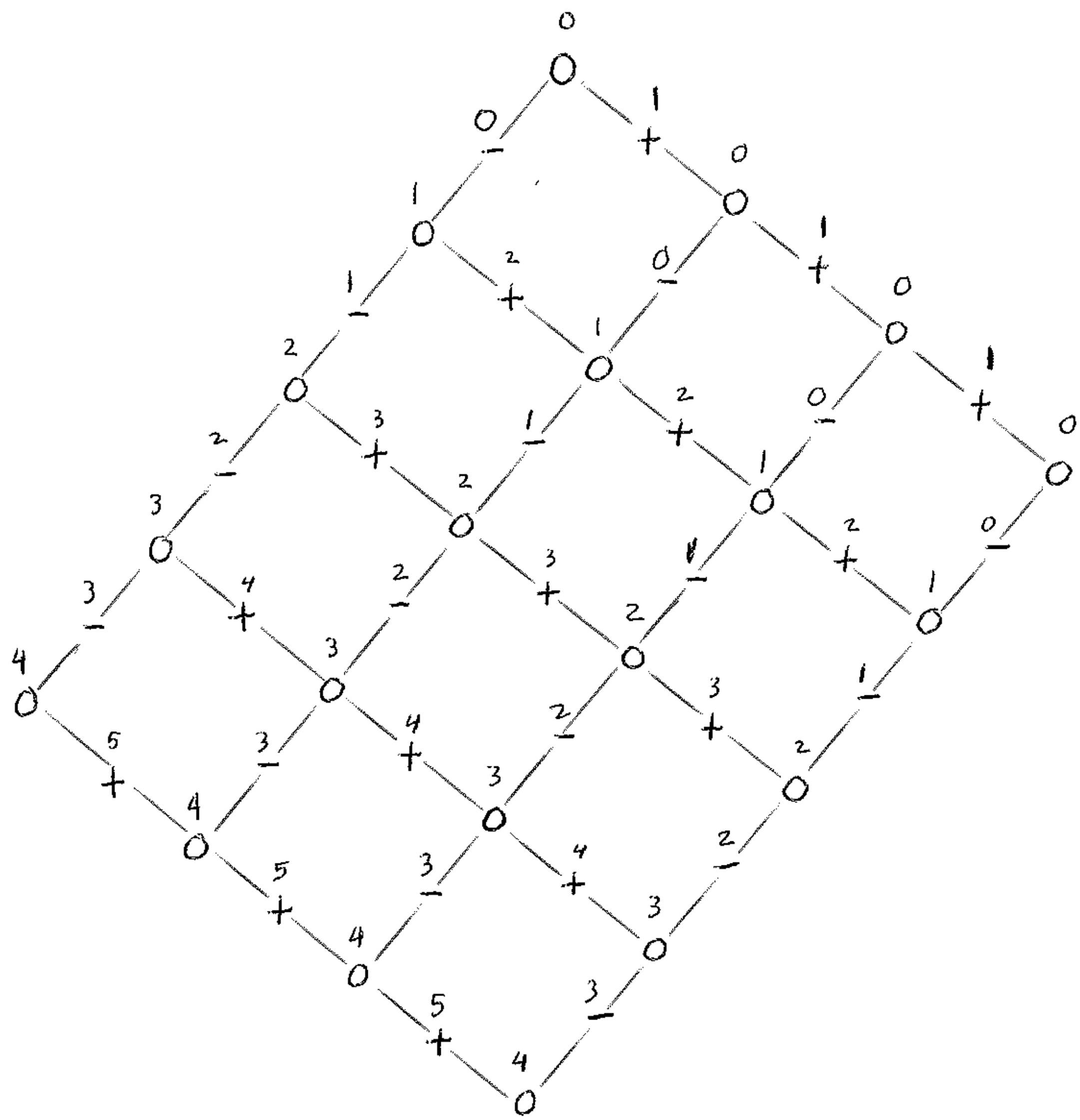
showing φ_2



$\mathfrak{F} = A_2, \quad GL(3), \quad B_\lambda, \quad \lambda = (7,3)$

subcrystal induced on levels $-1, 0, 1$

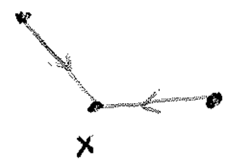
showing ε_2



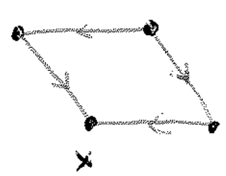
$\Phi = A_2$, $\quad GL(3)$, $\quad B_\lambda$, $\quad \lambda = (7, 3)$

Observations

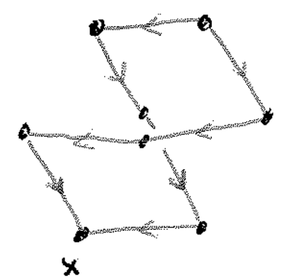
(i) Given



then either



(if x has level $\neq 0$)

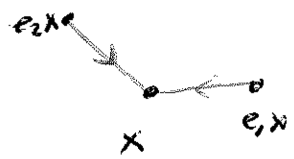


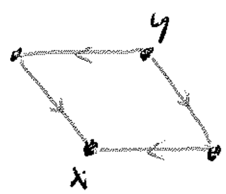
(if x has level 0)

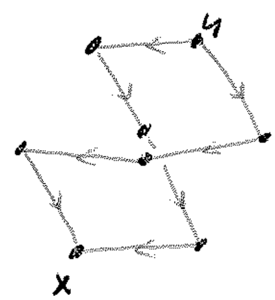
$\Phi = A_2$, $\quad GL(3) \quad$ B_λ, $\quad \lambda = (7,3)$

Observations

(ii) Referring to (i), write







Then for all $z \in B_\lambda$,

$$e_1 x \preceq z \quad \text{and} \quad e_2 x \preceq z \quad \text{implies} \quad y \preceq z$$

In other words, partial order $\preceq$ makes B_λ a lattice.