Lecture 15 Mon Oct 7

10/7/19

Crystals and Schur functions

For our crystal B_λ consider the character

$$\chi(B_\lambda) = \sum_{T \in B_\lambda} t^{wt(T)} \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

For $T \in B_\lambda$,

$$t^{wt(T)} = t_1^{\mu_1} t_2^{\mu_2} \cdots t_n^{\mu_n}$$

μ_i = # boxes in T that contain i

Above $\chi(B_\lambda)$ is denoted

$s_\lambda(t_1, \ldots, t_n)$ "Schur polynomial"

We saw earlier that $\chi(B_\lambda)$ is invariant under the Weyl group $W = S_n$.

this shows that the poly s_λ is symmetric

Ex For $n=3$ and $\lambda = (2,1)$

$s_\lambda(t_1, t_2, t_3)$ = sum of

$t_1 t_2^2$ $t_1^2 t_2$

$t_2^2 t_3$ $2 t_1 t_2 t_3$ $t_1^2 t_3$

$t_2 t_3^2$ $t_1 t_3^2$

—— o ——

Recall elem sym functions (for $n=3$)

$e_1 = t_1 + t_2 + t_3$

$e_2 = t_1 t_2 + t_1 t_3 + t_2 t_3$

$e_3 = t_1 t_2 t_3$

For above s_λ,

$s_\lambda = e_1 e_2 - e_3$ *

Recall complete sym functions

Compl sym function	sum of
h_1	$t_2 \quad t_1$ t_3
h_2	$t_2^2 \quad t_1t_2 \quad t_1^2$ $t_2t_3 \quad t_1t_3$ t_3^2
h_3	$t_2^3 \quad t_1t_2^2 \quad t_1^2t_2 \quad t_1^3$ $t_2^2t_3 \quad t_1t_2t_3 \quad t_1^2t_3$ $t_2t_3^2 \quad t_1t_3^2$ t_3^3

For above s_λ,

$$s_\lambda = h_1 h_2 - h_3$$

KK

Interp of *, **

One checks

$$B_{\square} \otimes B_{(1,1)} = B_{(2,1)} \cup B_{(1,1,1)}$$

So

$$\begin{array}{ccccc} s_{\square} & s_{(1,1)} & = & s_{(2,1)} & + & s_{(1,1,1)} \\ \| & \| & & \| & & \| \\ e_1 & e_2 & & s_\lambda & & e_3 \end{array}$$

One checks

$$B_{\square} \otimes B_{(2)} = B_{(2,1)} \cup B_{(3)}$$

So

$$\begin{array}{ccccc} s_{\square} & s_{(2)} & = & s_{(2,1)} & + & s_{(3)} \\ \| & \| & & \| & & \| \\ h_1 & h_2 & & s_\lambda & & h_3 \end{array}$$

In fact *, ** hold for all n (ex 3.3)

Later we will use products of schur functions to describe how a tensor product of crystals decomposes into connected components.

$I = A_r$ $\quad$ Λ type $GL(r+1)$ $\quad$ $n = r+1$

Fix $k \geq 1$

For a partition $\lambda \vdash k$ of length $\leq n$

the crystal B_λ is isomorphic to a connected component of $B^{\otimes k}$ $\quad$ $B = B(1)$

Reverse logical direction.

Consider the connected components of $B^{\otimes k}$

We will see:

- For a connected component C of $B^{\otimes k}$, C has unique hw vector
- the hw λ of C is a partition of k with length $\leq n$
- crystals C, B_λ are iso
- $\#$ hw vectors in $B^{\otimes k}$ of wt λ $=$ $\#$ standard tableaux of shp λ

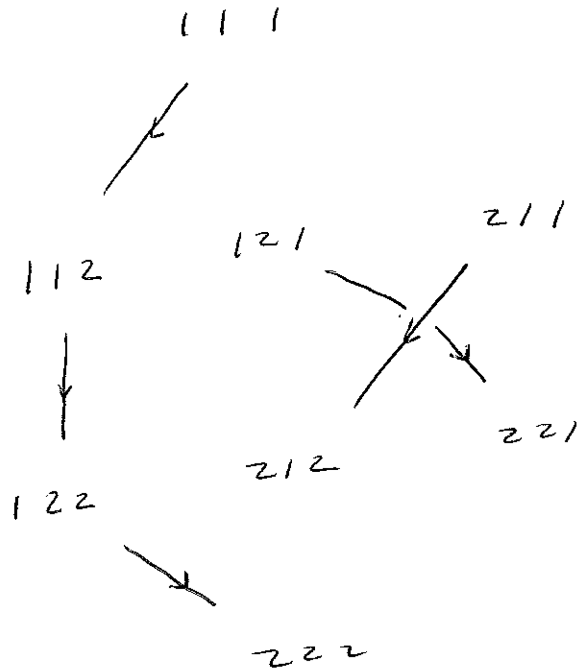
$\longrightarrow$ next goal

Ex $r=1$, $k=3$, $n=2$

B: $\boxed{2} \leftarrow \boxed{1}$

$\leftarrow$ means $\xleftarrow{1}$

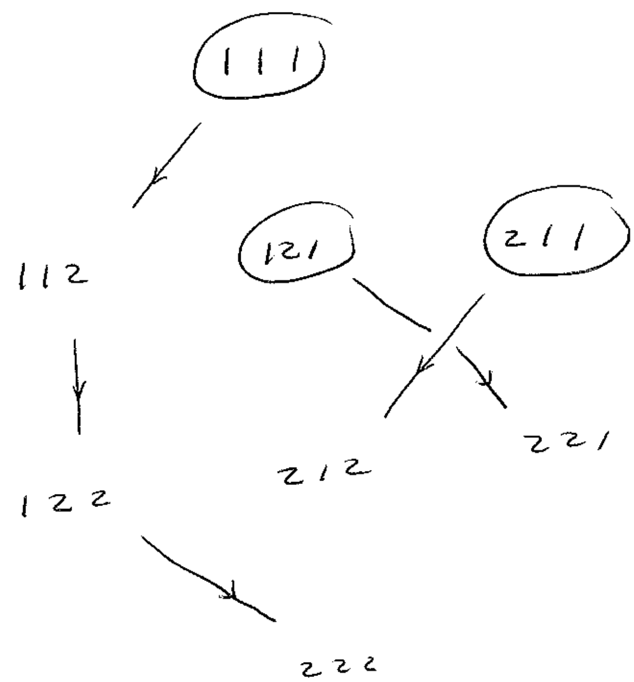
$B^{\otimes 3}$:

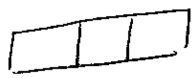
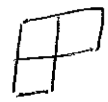


abc means

$\boxed{a} \otimes \boxed{b} \otimes \boxed{c}$

hw vectors circled



partitions $\lambda \vdash 3$ of length ≤ 2	$(3,0)$	$(2,1)$
$YD(\lambda)$		
hw vectors in $B^{\otimes 3}$ with wt λ	111	121, 211
# hw vectors in $B^{\otimes 3}$ with wt λ	1	2
stand tableaux of shp λ	1 2 3	1 2 / 3 , 1 3 / 2
# stand. tableaux of shp λ	1	2

Back to general r, k

Describe hw vectors in $B^{\otimes k}$

For $x \in B^{\otimes k}$

find nec/suf conditions for x to be hw

Write
$$wt(x) = \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

Write
$$X = \boxed{a_k} \otimes \boxed{a_{k-1}} \otimes \cdots \otimes \boxed{a_2} \otimes \boxed{a_1}$$
$a_i \in \{1, 2, \dots, n\}$

For $1 \le i \le r$ apply signature rule $\overset{i \; i+1}{)\,(}$ to x

and cancel $()$ pairs

Get
$$\underbrace{)) \cdots)}_{\varphi_i(x)} \; \underbrace{((\cdots (}_{\varepsilon_i(x)}$$

Consider when is $\varepsilon_i(x) = 0$

Consider set
$$\{ r \mid 1 \le r \le k, \; a_r = i \}$$
$$= \{ r_1, r_2, \dots, r_{\lambda_i} \} \qquad r_1 < r_2 < \cdots$$

Consider set

$$\{ s \mid 1 \le s \le k,\ a_s = i+1 \}$$

$$= \{ s_1, s_2, \ldots, s_{\lambda_{i+1}} \} \qquad s_1 < s_2 < \cdots$$

<u>Claim</u> TFAE

(i) $\varepsilon_i(x) = 0$

(ii) $\lambda_i \ge \lambda_{i+1}$ and

$$r_\ell < s_\ell \qquad 1 \le \ell \le \lambda_{i+1}$$

<u>pf cl</u> (i) $\longrightarrow$ (ii)

After signature rule $\begin{matrix} i & i+1 \\ ) & (\end{matrix}$ each (is cancelled.

So

$$\begin{matrix} \#\ (\text{'s} & \le & \#\ )\text{'s} \\ \parallel & & \parallel \\ \lambda_{i+1} & & \lambda_i \end{matrix}$$

Also for $1 \le \ell \le \lambda_{i+1}$ consider location s_ℓ in x

$$x: \quad \boxed{a_k} \otimes \cdots \otimes \underset{\underset{s_\ell}{\uparrow}}{\underset{(}{\boxed{i+1}}} \otimes \cdots \otimes \underset{\underset{s_{\ell-1}}{\uparrow}}{\underset{(}{\boxed{i+1}}} \otimes \cdots\ \text{say}\ \cdots \otimes \underset{\underset{s_1}{\uparrow}}{\underset{(}{\boxed{i+1}}} \otimes \cdots \otimes \boxed{a_1}$$

For each (to cancel, must exist at least ℓ copies of) to right of s_ℓ

So
$$r_\ell < s_\ell$$

(ii) → (i) Routine.

claim proved ✓

Using x we create an array $\varphi = \varphi(x)$ of boxes, as follows

φ has k boxes and n rows

For $1 \le i \le n$
row i of φ gets d_i boxes

Fill row i of φ with
$$\{ r \mid 1 \le r \le k,\ a_r = i \}$$
In inc order $< < < \cdots$

LEM With above notation TFAE

(i) x is hw

(ii) $\lambda \vdash k$ and

Q is standard tableau shp λ

pf By claim.

COR For a partition $\lambda \vdash k$ of length $\leq n$

the map $x \longrightarrow Q(x)$

is a bijection from

(i) the set of hw vectors in $B^{\otimes k}$ of wt λ

to (ii) the set of stand tableaux shp λ

COR For a partition $\lambda \vdash k$ of length $\leq n$

\# hw vectors in $B^{\otimes k}$ of wt λ

=

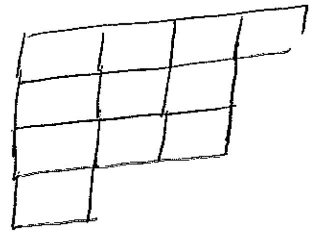
\# standard tableaux of shp λ. □

In the next example we illustrate the bijection $x \longleftrightarrow \varphi(x)$

$k = 11$

Ex $\lambda = (4, 3, 3, 1)$

$YD(\lambda)$:



$x =$

$$\boxed{3} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{1}$$

11	10	9	8	7	6	5	4	3	2	1

$\varphi(x) =$

1	2	5	6
3	4	7	
8	9	11	
10			

Show X is hw

Apply sig $\overset{1}{)}\overset{2}{(}$

3 4 3 3 2 1 1 2 2 1 1

( )) (( ))

$\varphi_1(x) = 1, \quad \varepsilon_1(x) = 0$ ✓

Apply sig $\overset{2}{)}\overset{3}{(}$

3 4 3 3 2 1 1 2 2 1 1

((( )))

$\varphi_2(x) = 0, \quad \varepsilon_2(x) = 0$ ✓

Apply sig $\overset{3}{)}\overset{4}{(}$

3 4 3 3 2 1 1 2 2 1 1

) ( ))

$\varphi_3(x) = 2, \quad \varepsilon_3(x) = 0$ ✓

□