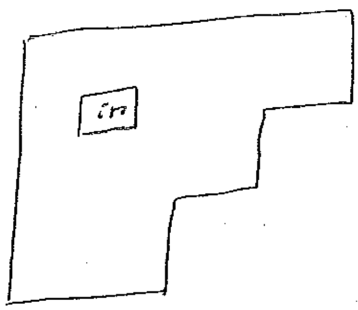
$(ii) \to (i)$ $\quad \exists\ T \in B_\lambda$ such that

$$\mathrm{RowR}(T) = y.$$

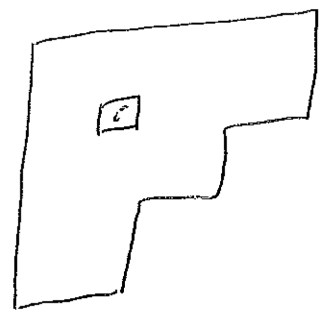
X is obtained from y by replacing single $\boxed{i}$ by $\boxed{i+1}$.

Back in T, replace corresp $\boxed{i}$ by $\boxed{i+1}$ to get S

S :



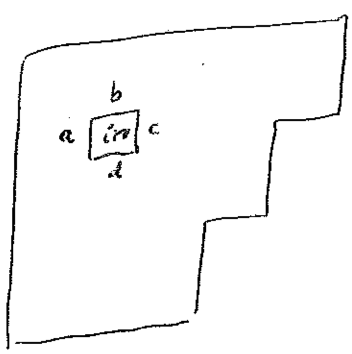
T :



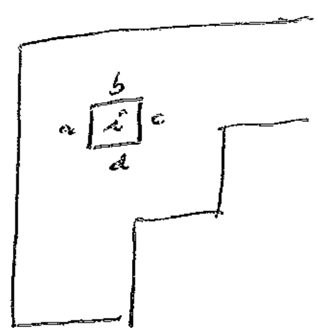
show S is semi standard.

Consider

S:



T:



Require.

$a \le i+1 \le c$

$b < i+1 < d$

Have

$a \le i \le c$

$b < i < d$

ok unless $\quad c = i \quad$ or $\quad d = i+1$

Show $c \neq i$

Suppose $c = i$

$T \xrightarrow{RowR}$ $\quad -- \ \boxed{i} \ i \ ---$

Apply signature rule $\underset{)}{i} \ \underset{(}{i+1}$

$-- \ \boxed{i} \ i \ ---$
$-- \ \ ) \ \ ) \ \ --$

After cancelling $()$ pairs, $\underset{)}{\boxed{i}}$ is rightmost surviving $)$

So $\underset{)}{i}$ was cancelled by some $\underset{(}{i+1}$ to right of $\underset{)}{\boxed{i}}$

No room for such $\underset{(}{i+1}$, contr.

Show $d \neq i+1$

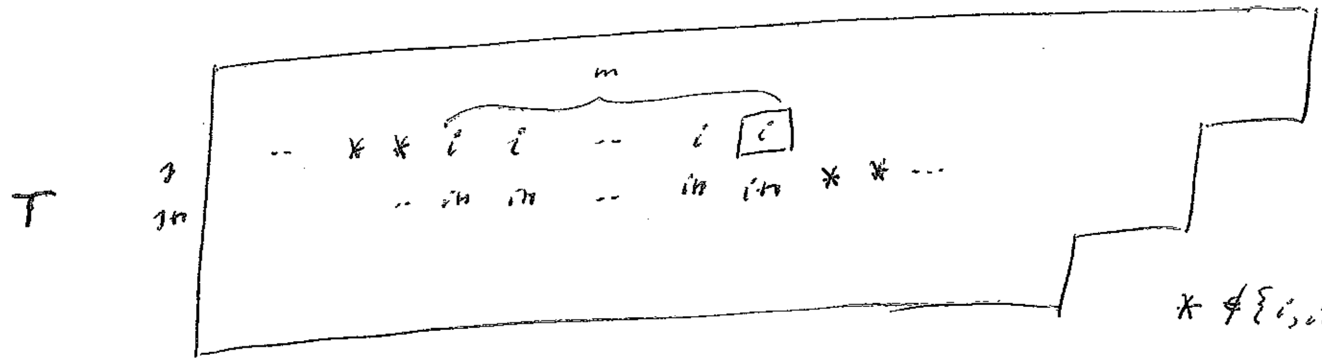
assume $d = i+1$

Consider row of T that contains $\boxed{i}$
$\quad\quad\quad\quad\quad\quad\ \ \ ''$
$\quad\quad\quad\quad\quad\quad\ \ \ j$

Let m = # boxes in row j of T that contain i

So $m \geq 1$.

Must have

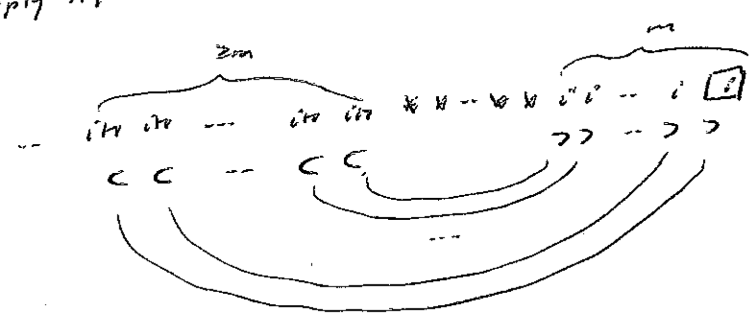


$* \notin \{i, i+1\}$

In row $j+1$ of T, at least m boxes contain $i+1$

$$T \xrightarrow{\text{RowR}} \cdots \underbrace{i{+}1\ i{+}1 \cdots i{+}1\ i{+}1}_{2m}\ * * \cdots * *\ \underbrace{i\ i \cdots i\ \boxed{i}}_{m}$$

Apply signature rule $\begin{matrix} i & i+1 \\ ) & (\end{matrix}$ and cancel $()$ pairs



$\underset{)}{\boxed{i}}$ is cancelled.

But after cancelling, $\underset{)}{\boxed{i}}$ is right most surviving $)$,
contr.

$\square$

LEM $\quad RowR(B_\lambda)$ is a connected component of the crystal $B^{\otimes|\lambda|}$

pf By LEM, $RowR(B_\lambda)$ is disjoint union of connected components of $B^{\otimes|\lambda|}$.

Each connected component of $B^{\otimes|\lambda|}$ contains at least one hw vector.

By LEM $\quad RowR(B_\lambda)$ contains unique hw vector.

Result follows $\square$

We saw that $\mathrm{RowR}(\mathcal{B}_\lambda)$ is a connected component of $\mathcal{B}^{\otimes|\lambda|}$, and hence a full subcrystal of the crystal $\mathcal{B}^{\otimes|\lambda|}$.

For this subcrystal, pull back the crystal structure to $\mathcal{B}_\lambda$ via RowR.

This turns $\mathcal{B}_\lambda$ into a crystal for Φ, Λ and desired wt function.

Crystal $\mathcal{B}_\lambda$ is seminormal since $\mathcal{B}^{\otimes|\lambda|}$ is seminormal.

By construction,

- the function
$$\mathrm{RowR}: \quad \mathcal{B}_\lambda \longrightarrow \mathcal{B}^{\otimes|\lambda|}$$
is an injective crystal morphism

- $\mathcal{B}_\lambda$ has a unique hw vector u_λ.

- $\mathcal{B}_\lambda$ is connected

So far, we used RowR to give B_λ a crystal structure.

Instead of RowR, we could have used ColR.

ColR gives the same crystal str on B_λ, as we now show.

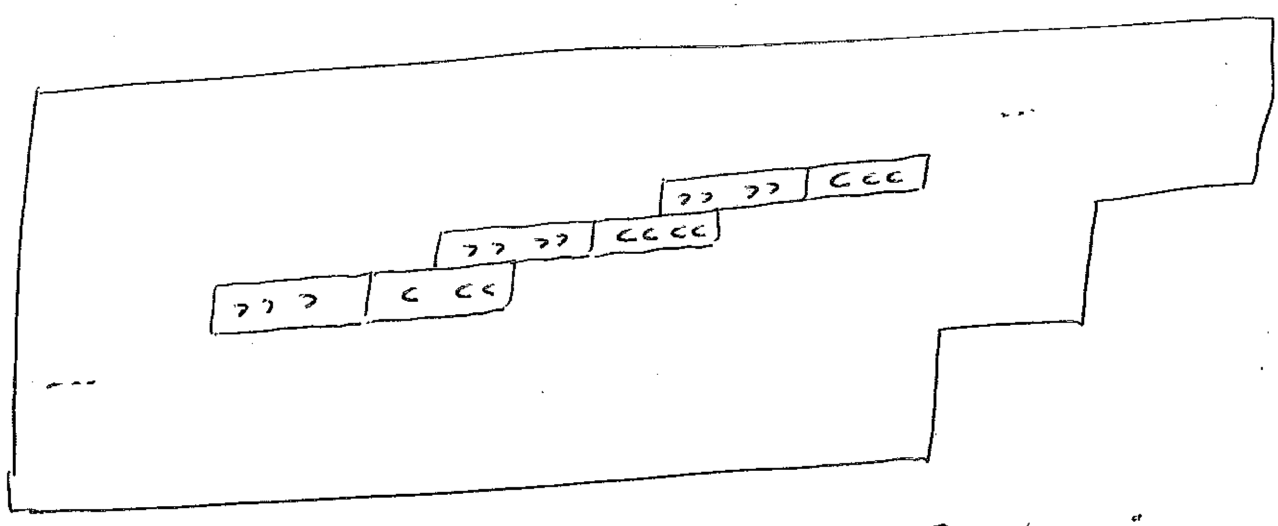
Given $T \in B_\lambda$

Compare RowR(T), ColR(T)

Given $1 \le i \le r$

Consider boxes in T that contain i or $i+1$

T:



key $\boxed{?}$ contains i

$\boxed{c}$ contains $i+1$

A column in T is <u>i-matched</u> whenever it contains $\boxed{i}$ and $\boxed{i+1}$

A box in T is i-matched whenever it is contained in an i-matched column

Apply RowR to T

Apply signature rule $\overset{i}{)}\ \overset{i+1}{(}$ to RowR(T), and cancel $()$ pairs

All i-matched boxes are cancelled.

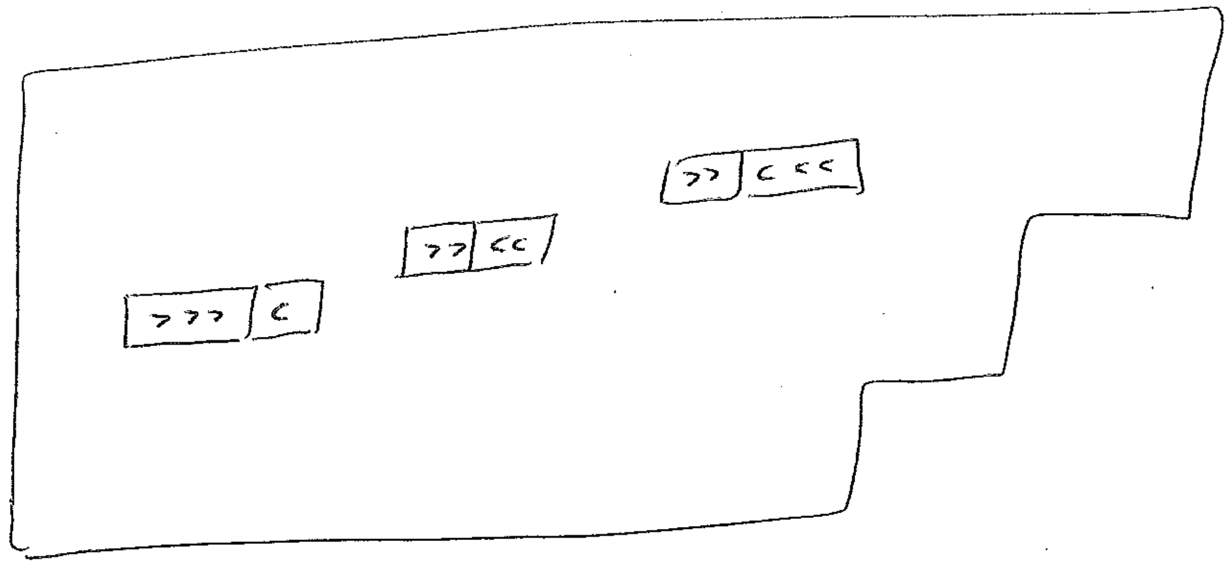
Repeat procedure using ColR

Again, all i-matched boxes are cancelled.

Now consider

boxes in T that contain i or $i+1$, but are not i-matched

T:



showing ☆

Row R, Col R put boxes ☆ in same order

So for Row R, Col R the signature rule $\overset{i}{>}\overset{in}{<}$

gives same outcome

This shows that Row R, Col R give B_λ the same

crystal structure.

Ex For $n=4$, $\lambda = (4,3,1)$

show that in B_λ,

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \xleftarrow{\ 1\ } \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

Sol. I Apply RowR

$$3\ 2\ 3\ 3\ \ 1\ 2\ 2\ 3 \xleftarrow[1]{?} 3\ 2\ 3\ 3\ \ 1\ 1\ 2\ 3$$

Apply signature rule $\overset{1}{)}\ \overset{2}{(}$ and cancel ( ) pairs

$$\begin{array}{cccccccc} 3 & 2 & 3 & 3 & 1 & 2 & 2 & 3 \\ & (& & &) & (& (& \end{array} \xleftarrow[1]{?} \begin{array}{cccccccc} 3 & 2 & 3 & 3 & 1 & 1 & 2 & 3 \\ & (& & &) &) & (& \end{array}$$

rightmost surviving)
change it to (

ok

Sol II Apply Col R

$$3\,2\,1\,3\,2\,3\,2\,3 \xleftarrow[1]{?} 3\,2\,1\,3\,1\,3\,2\,3$$

Apply signature rule $\overset{1}{)}\overset{2}{(}$ and cancel $()$ pairs

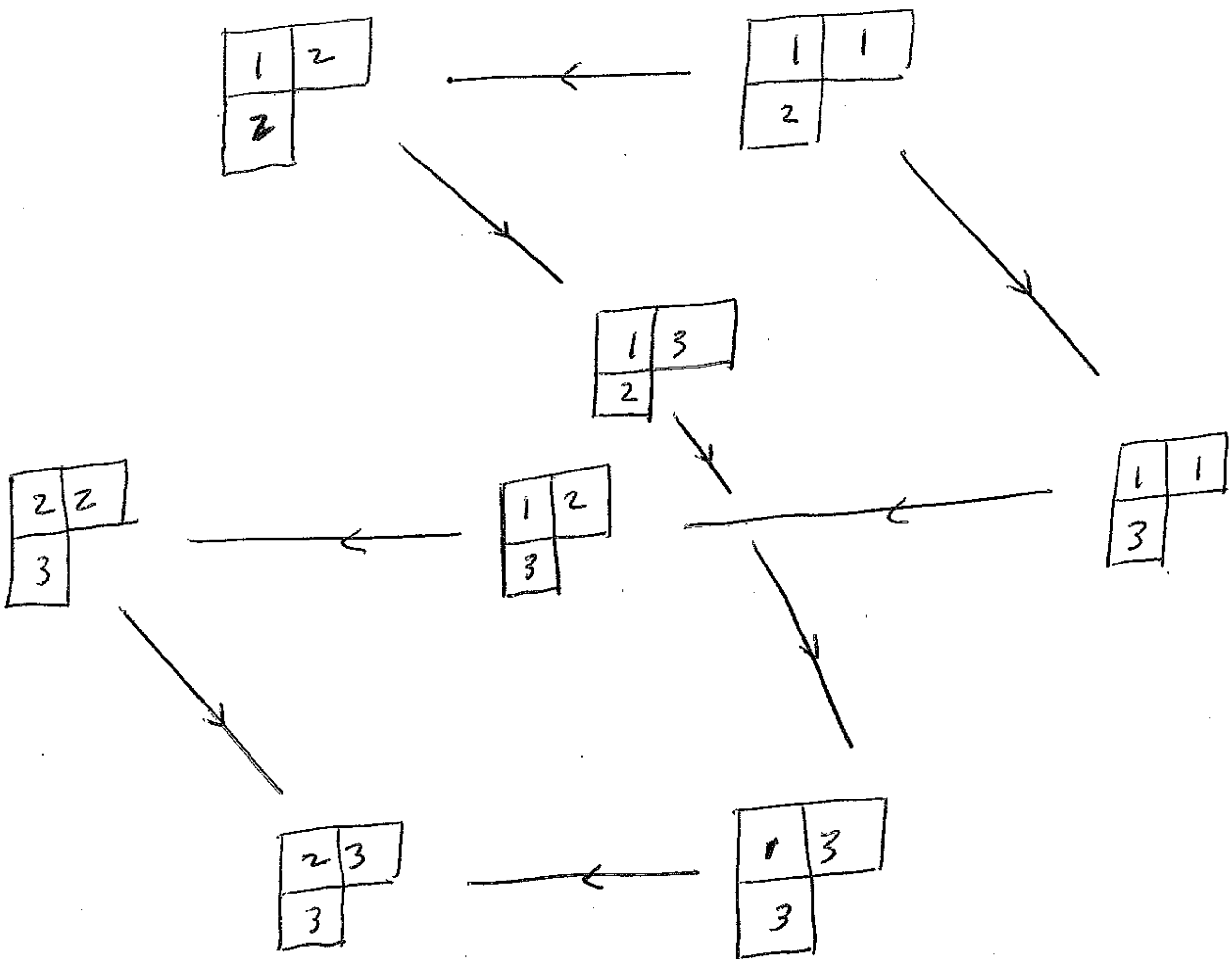
$$\begin{array}{cccccccc} 3&2&1&3&2&3&2&3\\ &(&)&&(&&(& \end{array} \xleftarrow[1]{?} \begin{array}{cccccccc} 3&2&1&3&1&3&2&3\\ &(&)&&)&&(& \end{array}$$

right-most surviving $)$
change it to $($

OK

$\square$

Ex For $n = 3$ and $\lambda = (2,1)$ the crystal graph for B_λ is



key