Lecture 13 Wednesday Oct 2

Next goals

- Turn B_λ into a semi normal crystal for Φ, Λ and above wt function.

- Display an injective crystal morphism

  $B = B_{(1)}$

  $$\mathrm{Row}R : \quad B_\lambda \longrightarrow B^{\otimes|\lambda|}$$

  "row reading"

- Display an injective crystal morphism

  $$\mathrm{Col}R : \quad B_\lambda \longrightarrow B^{\otimes|\lambda|}$$

  "column reading"

- Show crystal B_λ has unique hw vector u_λ

- Show crystal B_λ is connected

<u>Note</u> For $\lambda = (k)$ and $\lambda = (1^k)$ we defined the crystal B_λ earlier

DEF For the partition λ.

we define functions

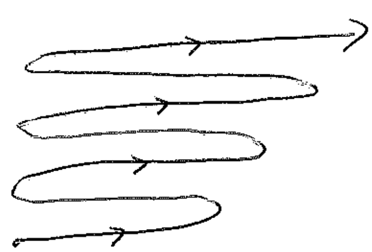
$$\mathrm{RowR}: \quad B_\lambda \longrightarrow B^{\otimes|\lambda|} \qquad B = B(1)$$

$$\mathrm{ColR}: \quad B_\lambda \longrightarrow B^{\otimes|\lambda|}$$

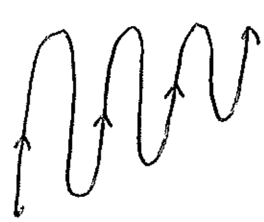
as follows.

For $T \in B_\lambda$,

RowR puts the entries of T in one row, in order



ColR puts the entries of T in one row, in order



The functions RowR, ColR are injective.

Ex For $\lambda = (3, 2)$

Row R sends

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array} \longrightarrow \boxed{d} \otimes \boxed{e} \otimes \boxed{a} \otimes \boxed{b} \otimes \boxed{c}$$

Col R sends

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array} \longrightarrow \boxed{d} \otimes \boxed{a} \otimes \boxed{e} \otimes \boxed{b} \otimes \boxed{c}$$

LEM For the partition λ and $T \in B_\lambda$,

$$wt(T) = wt(\mathrm{RowR}(T))$$

$$wt(T) = wt(\mathrm{ColR}(T))$$

Pf Write $k = |\lambda|$. Recall, for

$$b = \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_k} \quad \in \quad B^{\otimes k}$$

we have

$$wt(b) = \sum_{i=1}^{k} wt\left(\boxed{a_i}\right)$$

and $wt\left(\boxed{a}\right) = e_a$ $\square$

Strategy

(i) show $\mathrm{RowR}(\mathcal{B}_\lambda)$ is a connected component of the crystal $\mathcal{B}^{\otimes|\lambda|}$

(ii) show $\mathrm{ColR}(\mathcal{B}_\lambda)$ is a connected component of the crystal $\mathcal{B}^{\otimes|\lambda|}$

(iii) For the connected component in (i), pull back the crystal structure to $\mathcal{B}_\lambda$ via RowR, this turns $\mathcal{B}_\lambda$ into a crystal for Φ, Λ with desired wt functions

(iv) By constr
$$\mathrm{RowR} : \mathcal{B}_\lambda \to \mathcal{B}^{\otimes|\lambda|}$$
is a crystal morphism.

(v) show
$$\mathrm{ColR} : \mathcal{B}_\lambda \to \mathcal{B}^{\otimes|\lambda|}$$
is a crystal morphism.

LEM For $T \in \mathcal{B}_\lambda$ and $x = \mathrm{RowR}(T)$,

TFAE

(i) $T = u_\lambda$

(ii) x is hw vectr in $\mathcal{B}^{\otimes|\lambda|}$

pf (i) $\to$ (ii)

Fr $1 \le i \le r$ show

$$\varepsilon_i(x) = 0$$

$$x = \cdots ** \cdots * \overbrace{i{+}1\ i{+}1 \cdots i{+}1\ i{+}1}^{\lambda_{i+1}}\ \overbrace{i\ i \cdots i\ i}^{\lambda_i}\ ** \cdots$$

$* \notin \{i, i{+}1\}$

Apply signature rule $\overset{i}{>}\ \overset{i+1}{<}$ and cancel $<>$ pairs

$$\begin{array}{cccccccccccccc} ** \cdots * & i{+}1 & i{+}1 & \cdots & i{+}1 & i{+}1 & i & i & \cdots & i & i & ** \cdots \\ & < & < & \cdots & < & < & > & > & \cdots & > & > & \end{array}$$

Every $<$ is cancelled since $\lambda_i \ge \lambda_{i+1}$

$\varphi_i(x) = \lambda_i - \lambda_{i+1}$, $\varepsilon_i(x) = 0$ ✓

(ii) → (i)   Write l = length(λ)

For $1 \le i \le l$ show

each box in row i of T contains i $*$

Suppose $*$ fails.   WLOG i minimal st $*$ fails.

Let j = entry in last box of row i of T

Each entry in row i of T is at least i and at most j.

$j > i$ since $*$ fails.

$$x: \quad \overbrace{\cdots\; j}^{\lambda_i}\; \overbrace{i{+}1\; i{+}1 \cdots i{+}1}^{\lambda_{i+1}}\; \overbrace{i{+}2\; i{+}2 \cdots i{+}2}^{\lambda_{i+2}} \cdots$$

Apply signature rule $\begin{smallmatrix} j-1 & j \\ ) & (\end{smallmatrix}$ and cancel ( ) pairs

$$\begin{array}{l} \cdots\; j \;\; i{+}1\; i{+}1 \cdots i{+}1\; i{+}2\; i{+}2 \cdots \\ \;\;\cdots\; (\\ \;\;\quad\;\uparrow \\ \text{does not cancel} \end{array}$$

So $\varepsilon_{j-1}(x) \ge 1$.

Now x not hw, cont. $\square$

LEM For $x, y \in B^{\otimes|\lambda|}$ and $1 \le i \le r$

assume

$$x \xleftarrow{i} y$$

Then TFAE:

(i) $x \in \mathrm{RowR}(B_\lambda)$

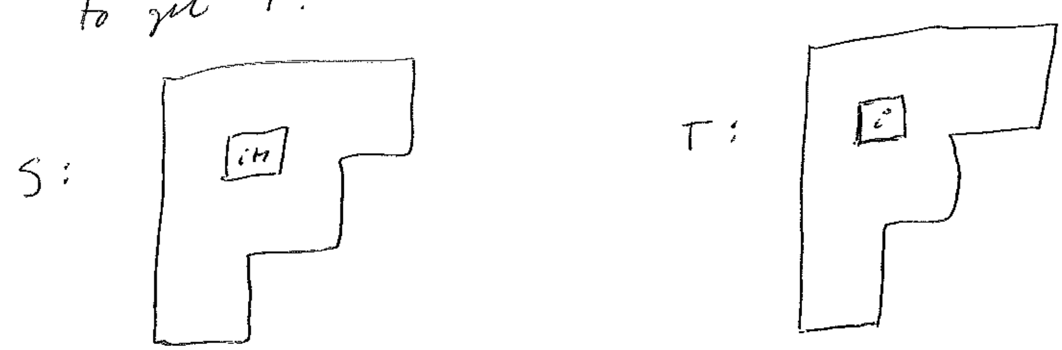
(ii) $y \in \mathrm{RowR}(B_\lambda)$

pf (i) $\to$ (ii) $\exists$ $S \in B_\lambda$ such that

$$\mathrm{RowR}(S) = x$$

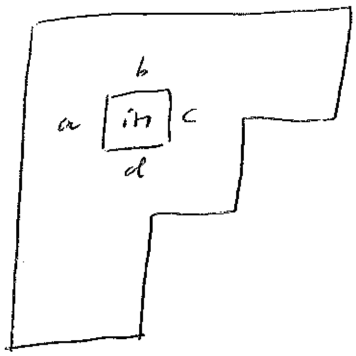
y is obtained from x by replacing single $\boxed{i+1}$ by $\boxed{i}$

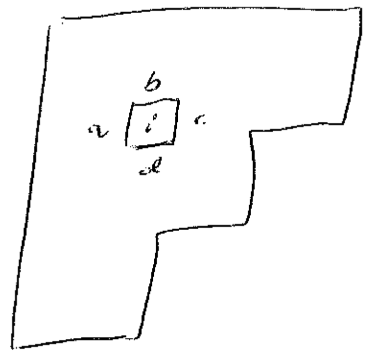
Back in S, replace corresponding $\boxed{i+1}$ by $\boxed{i}$ to get T.



show T is semi standard

Consider

S 

T: 

Have

$a \le i{+}1 \le c$

$b < i{+}1 < d$

Require

$a \le i \le c$

$b < i < d$

OK unless $a = i{+}1$ or $b = i$

show $a \ne i{+}1$

Suppose $a = i{+}1$

$S \xrightarrow{\text{RowR}} \cdots \; i{+}1 \; \boxed{i{+}1} \; \cdots$

apply signature rule $\quad \overset{i}{)} \; \overset{i+1}{(}$

$\cdots \; \underset{(}{i{+}1} \; \underset{(}{\boxed{i{+}1}} \; \cdots$

After cancelling () pairs, $\underset{(}{\boxed{i{+}1}}$ is left-most surviving (.

So $\underset{(}{i{+}1}$ was cancelled by some $\overset{i}{)}$ to left of $\underset{(}{\boxed{i{+}1}}$

No room for such $\overset{i}{)}$, contradiction.

Show $b \neq i$

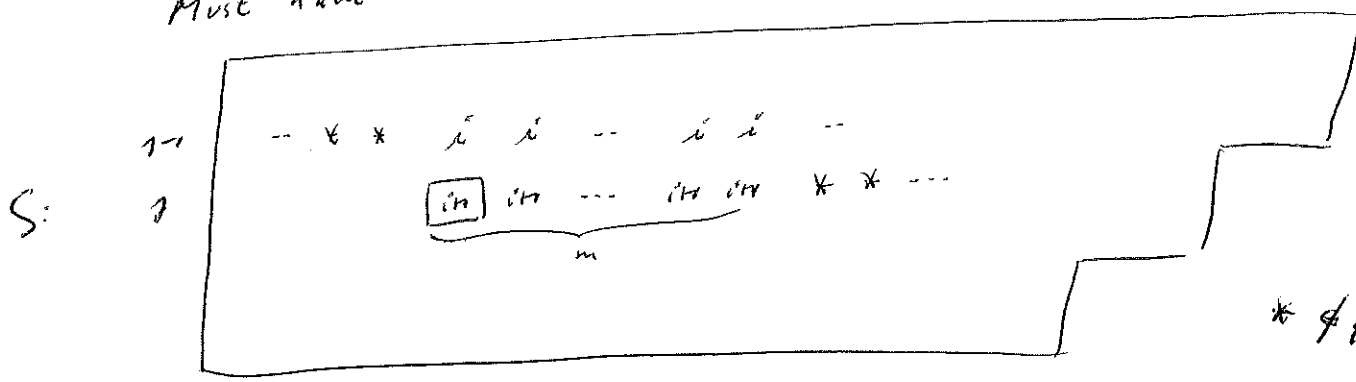
Assume $b = i$

Consider the $\underset{\substack{\| \\ j}}{\text{row}}$ of S that contains $\boxed{i+1}$

Let $m = \#$ boxes in row j of S that contain $i+1$

So $m \geq 1$

Must have



$* \notin \{i, i+1\}$

In row $j-1$ of S, at least m boxes contain i.

$$S \xrightarrow{\text{RowR}} \cdots \overbrace{\boxed{i+1}\ i+1 \cdots i+1\ i+1}^{m}\ * * \cdots * *\ \overbrace{i\ i \cdots i\ i}^{\geq m} \cdots$$

Apply signature rule $\underset{>}{i}\ \underset{<}{i+1}$ and cancel $<,>$ pairs

$$\cdots \overbrace{\underset{<}{\boxed{i+1}}\ \underset{<}{i+1} \cdots \underset{<}{i+1}\ \underset{<}{i+1}}^{m}\ * * \cdots * *\ \overbrace{\underset{>}{i}\ \underset{>}{i} \cdots \underset{>}{i}\ \underset{>}{i}}^{\geq m}$$

$\underset{<}{\boxed{i+1}}$ is cancelled.

But after cancelling $\underset{<}{\boxed{i+1}}$ is leftmost surviving $<$, contr.