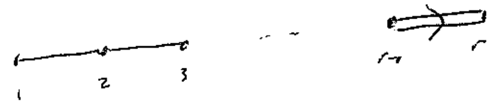
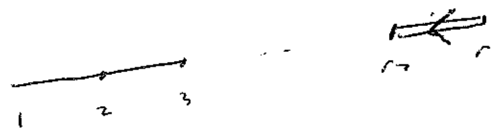
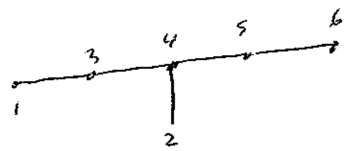
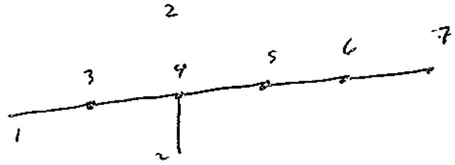
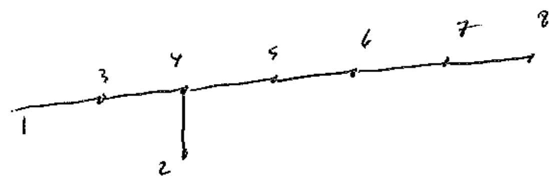
Lecture 12 Monday Sept 30

9/30/19

Φ	Dynkin diag
A_r	
B_r	
C_r	
D_r	
E_6	
E_7	
E_8	
F_4	
G_2	

Levi Root Systems

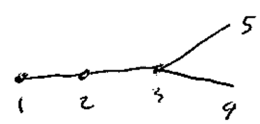
Given nonempty subset $J \subseteq I$

Define
$$\Sigma_J = \{\alpha_j\}_{j \in J}$$
$$\Phi_J = \Phi \cap \operatorname{Span}(\Sigma_J)$$
$$W_J = \text{subgroup of } W \text{ gen by } \{s_j\}_{j \in J}$$

Then
Φ_J is a root system with simple roots Σ_J — "Levi root system for J"
and Weyl group W_J

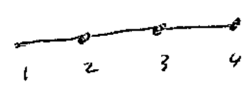
To obtain the Dynkin diag for Φ_J from the Dynkin diag for Φ, just delete all vertices except J

Ex $\Phi = D_5$



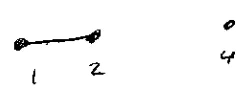
$J = \{1, 2, 3, 4\}$

$\Phi_J = A_4$



$J = \{1, 2, 4\}$

$\Phi_J = A_2 \times A_1$



orthog union of A_2 and A_1

Given a root system Φ with simple roots $\Sigma = \{\alpha_i\}_{i \in I}$ and wt lattice Λ

Given nonempty $J \subseteq I$. Observe

Λ is a weight lattice for Φ_J

Given crystal $\mathcal{B}$ with root data Φ, Λ

Using $\mathcal{B}$ we obtain a crystal $\mathcal{B}_J$ with root data Φ_J, Λ

$\mathcal{B}_J$ is obtained from $\mathcal{B}$ by forgetting

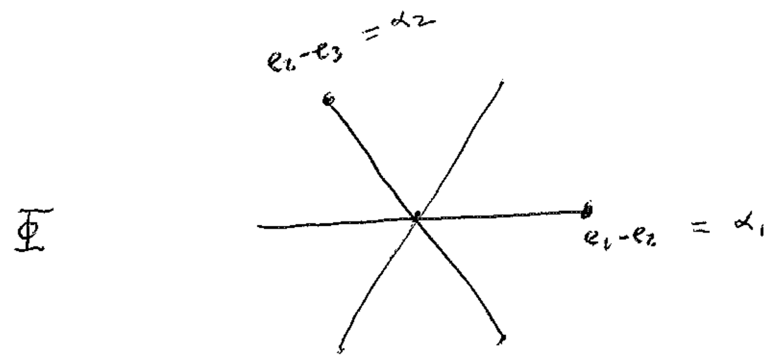
$f_i, e_i, \varphi_i, \varepsilon_i \quad i \in I \setminus J$

"Levi branched crystal"

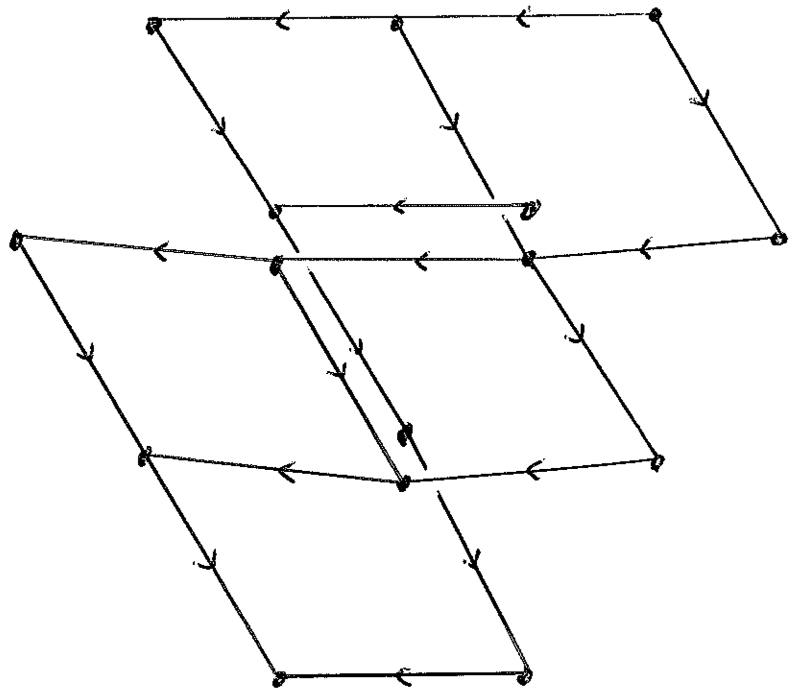
The crystal graph for $\mathcal{B}_J$ is obtained from the crystal graph for $\mathcal{B}$, by deleting all $\xleftarrow{i}$ for $i \in I \setminus J$.

Example Concerning Ex 2.6 (ii)

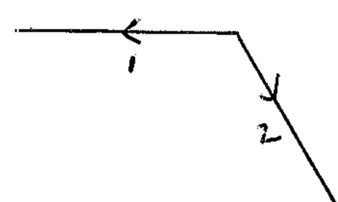
$\Phi = A_2$, Λ type $GL(3)$



B



Key



seminormal

Young tableaux

For $k \in \mathbb{N}$, a partition of k is a sequence of nonneg integers

$$\lambda = \{\lambda_i\}_{i=1}^{\infty}$$

such that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

and

$$k = \lambda_1 + \lambda_2 + \lambda_3 + \cdots$$

For above partition λ write

$$\lambda \vdash k \qquad \text{or} \qquad |\lambda| = k$$

The length of λ is

$$\ell = |\{i \mid \lambda_i > 0\}|$$

The parts of λ are

$$\lambda_1, \lambda_2, \ldots, \lambda_\ell$$

We usually suppress trailing 0's in the notation. For example write

$$\lambda = (4, 2, 2, 1)$$

for a partition of $k = 9$.

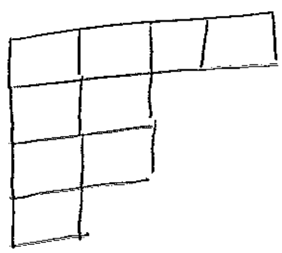
We represent any partition λ by its __Young diagram__ $YD(\lambda)$

also called __Ferrar diagram__ or __tableau shape__

Ex For $\lambda = (4, 2, 2, 1)$

\# boxes in row

$YD(\lambda)$:

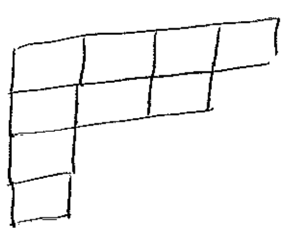


4
2
2
1

For a partition λ, its __conjugate partition__ λ' is obtained by swapping rows/cols in $YD(\lambda)$

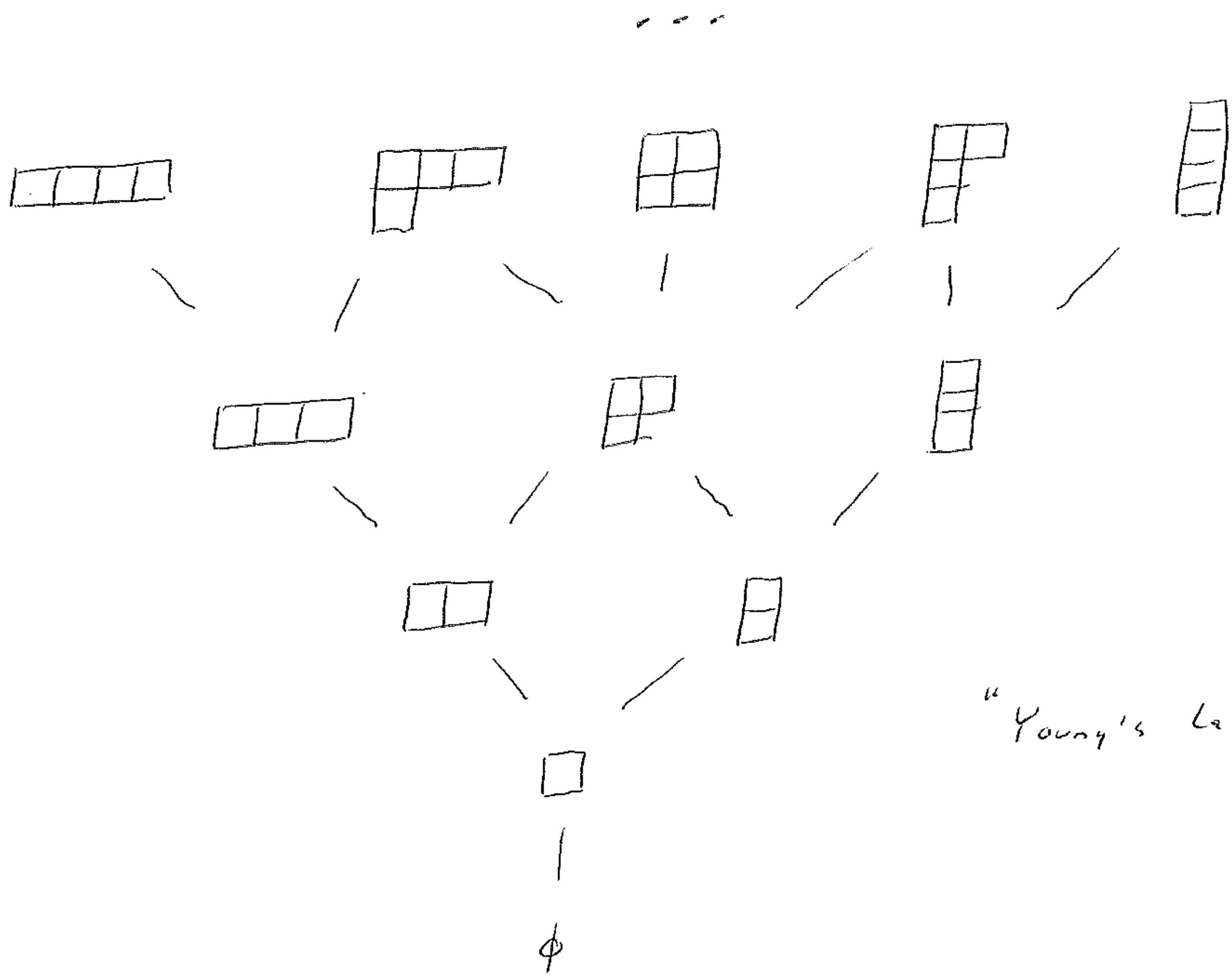
Ex Fn $\lambda = (4, 2, 2, 1)$

$YD(\lambda')$:



$\lambda' = (4, 3, 1, 1)$

The partitions form a poset



"Young's Lattice"

For partitions λ, μ

λ covers μ whenever $YD(\lambda)$ is obtained from $YD(\mu)$ by adding 1 box.

For a partition λ,

a <u>standard tableau of shape λ</u> is obtained from $YD(\lambda)$ by filling each box with an integer from $1, 2, \dots, |\lambda|$ such that

- in each row the numbers strictly increase $\longrightarrow$
- in each column the numbers strictly inc $\downarrow$

Ex For $\lambda = (3,1)$

$$YD(\lambda) = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}$$

the standard tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

<u>Note</u> For a partition λ the standard tableaux of shp λ are in bijection with the paths in Young's Lattice from $\emptyset$ up to λ

For a partition λ,

A <u>semistandard tableau of shape λ</u>

is obtained from $YD(\lambda)$ by filling each box

with a positive integer such that

- In each row the numbers weakly inc $\longrightarrow$
- In each column the numbers strictly inc $\downarrow$

Ex For $\lambda = (3, 1)$

Some semistandard tableaux are

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \cdots$$

For a partition λ

Define a semi standard tableau U_λ of shape λ such that

- For each row i, every box contains i

"Yamanouchi tableau of shp λ"

Ex Some Yamanouchi tableaux are

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

Until further notice fix integer $r \geq 1$.

Given root system $\Phi = A_r$

wt lattice Λ type $GL(r+1)$ $\qquad n = r+1$

Fix a partition λ of length $\leq n$

View $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda \subseteq \mathbb{R}^n = V$

DEF Let

B_λ = set of all semistandard tableaux of shape λ and each entry $\leq n$

DEF We define a function

$$wt : \begin{array}{ccc} B_\lambda & \longrightarrow & \Lambda \\ T & \longmapsto & \sum_{i=1}^{n} \mu_i e_i \end{array}$$

μ_i = # boxes of T that contain i

Ex For $n = 2$

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad wt(T) = e_1 + 3e_2$$

Ex $wt(u_\lambda) = \lambda$

Next goals

- Turn B_λ into a seminormal crystal for I, n and above wt function

- Display an injective crystal morphism

$B = B_{(1)}$

$$\text{RowR} : B_\lambda \longrightarrow B^{\otimes |\lambda|}$$

"row reading"

- Display an injective crystal morphism

"column reading"

$$\text{ColR} : B_\lambda \longrightarrow B^{\otimes |\lambda|}$$

- Show crystal B_λ has unique hw vector u_λ

- Show crystal B_λ is connected

Note For $\lambda = (k)$ and $\lambda = (1^k)$ we defined the crystal B_λ earlier