Lecture 11 Friday Sept 27 9/27/19

Note For $i \in I$ and $x, y \in B$

$$x \xleftarrow[i]{} y$$

iff

$$\sigma_i(y) \xleftarrow[i]{} \sigma_i(x)$$

However σ_i does not respect $\xleftarrow[j]{}$ for $j \neq i$

σ_i is not a crystal iso in general.

Note For distinct $i, j \in I$ recall the order $n(i,j)$ for $s_i s_j$. So

$$(s_i s_j)^{n(i,j)} = 1.$$

It turns out

$$(\sigma_i \sigma_j)^{n(i,j)} = 1,$$

provided that B has a property called "normal"

In this case, the Weyl group W of Φ acts on B st

$$s_i : \begin{array}{ccc} B & \to & B \\ x & \to & \sigma_i(x) \end{array} \qquad i \in I$$

the character of a crystal

Given a root system Φ and at lattice Λ

Recall $\Lambda, +$ is an abelian group.

Wish to discuss its group ring over $\mathbb{Z}$.

For notational convenience, turn additive group $\Lambda, +$

into a multiplicative group

$$\{t^\lambda\}_{\lambda\in\Lambda} \qquad\qquad t \text{ indeterminate}$$

with product

$$t^\lambda t^\mu = t^{\lambda+\mu} \qquad\qquad \lambda,\mu\in\Lambda$$

So

$$\begin{aligned} \Lambda &\longrightarrow \{t^\lambda\}_{\lambda\in\Lambda} \qquad\qquad * \\ \lambda &\longmapsto t^\lambda \end{aligned}$$

is iso of groups

View the group ring of Λ as

$$\mathcal{E} = \left\{ \sum_{\lambda\in\Lambda} a_\lambda t^\lambda \,\middle|\, a_\lambda\in\mathbb{Z}, \text{ all but fin many } a_\lambda \text{ are } 0 \right\}$$

Weyl group W of Φ acts on Λ

Iso $*$ induces a W-action on $\mathcal{E}$ as a group of ring automorphisms:

For $w \in W$ and $\lambda \in \Lambda$,

$$w(t^\lambda) = t^{w(\lambda)}$$

DEF For a finite crystal $\mathcal{B}$ with root data Φ, Λ

define
$$\chi(t) = \chi_{\mathcal{B}}(t) = \sum_{b \in \mathcal{B}} t^{wt(b)} \in \mathcal{E}$$

LEM For finite crystals $\mathcal{B}, \mathcal{B}'$ with root data Φ, Λ.

$$\chi_{\mathcal{B} \otimes \mathcal{B}'}(t) = \chi_{\mathcal{B}}(t)\, \chi_{\mathcal{B}'}(t)$$

pf Recall for $b \otimes b' \in \mathcal{B} \otimes \mathcal{B}'$

$$wt(b \otimes b') = wt(b) + wt(b')$$

$\square$

LEM For a finite semi normal crystal B
with root data Φ, Λ,

$\chi_B(t)$ is fixed by each $w \in W$.

pf

$$\chi_B(t) = \sum_{\lambda \in \Lambda} t^\lambda \left(\# \text{ elements in } B \text{ of wt } \lambda \right)$$

For $\lambda \in \Lambda$

$\#$ elements in B of wt λ

$= \quad \text{- - - -} \qquad$ wt $w(\lambda)$

Result follows. $\square$

Ex $\quad \Phi = A_r \quad$ type $GL(r+1) \quad n = r+1$

$$\Lambda = \sum_{i=1}^{n} \mathbb{Z} e_i$$

$$\mathcal{E} = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \qquad t_i = t^{e_i}$$

"Laurent polynomials in $t_1, \dots, t_n$"

Take $\quad \mathcal{B} = \mathcal{B}_{(1)}$

$$\begin{array}{lccccccccc} & \boxed{n} & \xleftarrow{n-1} & \boxed{n-1} & \leftarrow & \cdots & \xleftarrow{2} & \boxed{2} & \xleftarrow{1} & \boxed{1} \\ \text{wt} & e_n & & & & \cdots & & e_2 & & e_1 \end{array}$$

For $1 \le i \le n$,
$$\text{wt}\left(\boxed{i}\right) = e_i$$

So $\quad t^{\text{wt}(\boxed{i})} = t^{e_i} = t_i$

$$\chi_{\mathcal{B}}(t) = t_1 + t_2 + \cdots + t_n$$

For $k \geq 1$ take $\mathcal{B} = \mathcal{B}(k)$

vertices are

$$b = \begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_k \\ \hline \end{array} \qquad 1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n$$

$$wt(b) = \sum_{i=1}^{k} wt\left(\boxed{a_i}\right)$$

$$t^{wt(b)} = \prod_{i=1}^{k} t^{wt\left(\boxed{a_i}\right)} = \prod_{i=1}^{k} t_{a_i}$$

$$\chi_{\mathcal{B}}(t) = \sum_{1 \leq a_1 \leq \cdots \leq a_k \leq n} t_{a_1} t_{a_2} \cdots t_{a_k}$$

$=$ complete symmetric function $h_k = h_k(t_1, \ldots, t_n)$

For $k \geq 1$ take $\beta = B_{(1^k)}$

vertices are

$$\begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \qquad 1 \leq a_1 < a_2 < \cdots < a_k \leq n$$

$$X_\beta(t) = \sum_{1 \leq a_1 < \cdots < a_k \leq n} t_{a_1} t_{a_2} \cdots t_{a_k}$$

$=$ elem symmetric function $e_k = e_k(t_1, \ldots t_n)$

Isogeny

Given root system and weight lattice

$$\Phi, \quad \Lambda \qquad (*)$$

Given another pair

$$\Phi', \quad \Lambda' \qquad (**)$$

An isogeny from $*$ to $**$ is a

group hom $m: \Lambda \longrightarrow \Lambda'$

such that

- restriction $m|_{\Phi}: \Phi \to \Phi'$ is bij
- For $\lambda \in \Lambda$ and $\alpha \in \Phi$,
  $\langle \lambda, \alpha \rangle = \langle m\lambda, m\alpha \rangle$

Ex $\quad \Phi = A_r \quad \Lambda$ type $GL(r+1)$

$\Phi' = A_r \quad \Lambda'$ type $SL(r+1)$

$\Lambda' = \{ \lambda \in \Lambda \mid \langle \lambda, \delta \rangle = 0 \}$

$\delta = e_1 + \dots + e_n \qquad n = r+1$

Then

$$m: \Lambda \longrightarrow \Lambda'$$
$$e_i \longrightarrow e_i - \frac{\delta}{n} \qquad 1 \le i \le n$$

is an isogeny.

Ex $\quad \Phi = B_r \quad \Lambda$ type $SO(r+1)$

$\Lambda = \mathbb{Z}\Phi$

$\Phi' = B_r \quad \Lambda'$ type spin $(2r+1)$

$\Lambda \subseteq \Lambda'$

then $\quad m: \Lambda \longrightarrow \Lambda'$
$$\lambda \longrightarrow \lambda$$

is isogeny

LEM Given an isogeny m from ¥ to ¥¥

then for any crystal $\mathcal{B}$ with root data ¥

$\mathcal{B}$ becomes a crystal with root data ¥¥, using

same $e_i, f_i, \varphi_i, \varepsilon_i$ $i \in I$ and weight function

$$\mathcal{B} \xrightarrow[wt]{} \Lambda \xrightarrow[m]{} \Lambda'$$

pf Routine $\square$

Twisting

Given a crystal B with root data Φ, Λ

Given $\theta \in \Lambda$

then B becomes a crystal with root data

Φ, Λ and

$$wt'(x) = wt(x) + \theta \qquad x \in B$$

$$e_i' = e_i \qquad f_i' = f_i$$

$$\varphi_i'(x) = \varphi_i(x) + \langle \theta, \alpha_i^\vee \rangle \qquad i \in I$$

$$\varepsilon_i'(x) = \varepsilon_i(x)$$

" B twisted via θ "

this crystal is iso to $T_\theta \otimes B$ (ex)

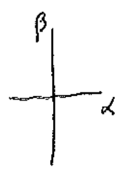
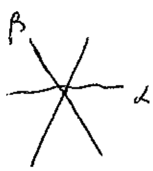
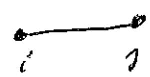
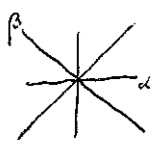
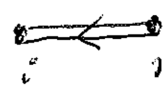
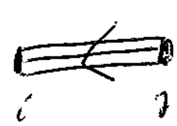
Dynkin diagram.

Given root system Φ with simple roots $\Sigma = \{\alpha_i\}_{i \in I}$

Corresponding Dynkin diagram has vertex set I.

For dist $i, j \in I$ connect i, j by a bond as follows.

write $\alpha = \alpha_i$ and $\beta = \alpha_j$.

Case	$n(i,j)$	bond
	2	
	3	
	4	
	6	

- arrow points from longer root to shorter root, $\#\text{bars} = \dfrac{\|\text{longer root}\|^2}{\|\text{shorter root}\|^2}$