

Signature Rule

Given semisimple crystals B_1, B_2, \dots, B_k with same root data.

Given $b_1 \otimes \dots \otimes b_k \in B_1 \otimes \dots \otimes B_k$ and $i \in I$
 \parallel
 x

Find $\varphi_i(x), \varepsilon_i(x), f_i(x), e_i(x)$

Attach to x a finite sequence involving 2 kinds of symbols

open parenthesis : \langle
 closed parenthesis : \rangle

F_n is $1 \leq j \leq k$, b_j contributes

$\underbrace{\rangle \rangle \dots \rangle \rangle}_{\varphi_i(b_j)} \underbrace{\langle \langle \dots \langle \langle}_{\varepsilon_i(b_j)}$

Cancel all matched pairs of adjacent parentheses $\langle \rangle$
 and iterate to obtain a sequence

$\rangle \rangle \dots \rangle \rangle \langle \langle \dots \langle \langle$

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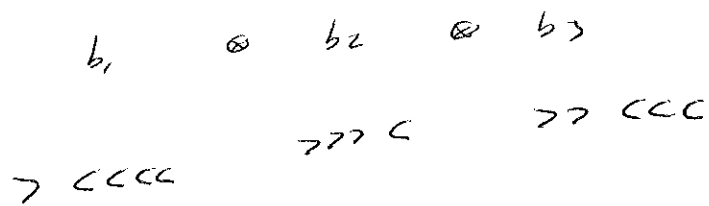
then

$\underbrace{\rangle \rangle \dots \rangle \rangle}_{\varphi_i(x)} \underbrace{\langle \langle \dots \langle \langle}_{\varepsilon_i(x)}$

ex $k=3$

$$x = b_1 \oplus b_2 \oplus b_3$$

b	b_1	b_2	b_3
$\varphi_i(b)$	1	3	2
$\varepsilon_i(b)$	4	1	3



$$\varphi_i(x) = 1, \quad \varepsilon_i(x) = 3$$

Referring to $x = b_1 \otimes \dots \otimes b_k$
 find $f_i(x)$, $e_i(x)$.

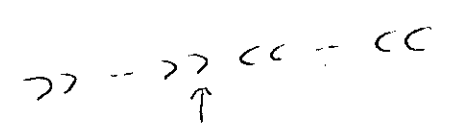
We have

$$f_i(x) = b_1 \otimes \dots \otimes f_i(b_t) \otimes \dots \otimes b_k$$

$$e_i(x) = b_1 \otimes \dots \otimes e_i(b_a) \otimes \dots \otimes b_k$$

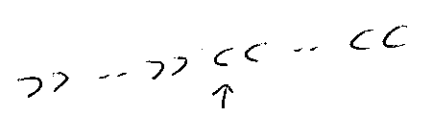
Find t, a :

b_t is attached to rightmost surviving $>$ in $*$:



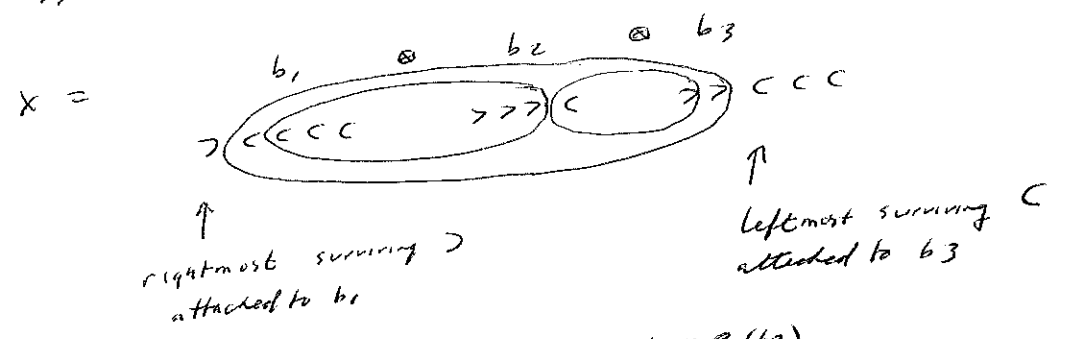
If no such $>$ then $f_i(x) = \emptyset$

b_a is attached to leftmost surviving $<$ in $*$:



If no such $<$ then $e_i(x) = \emptyset$

Ex $k=3$, revisited



$$f_i(x) = f_i(b_1) \otimes b_2 \otimes b_3$$

$$e_i(x) = b_1 \otimes b_2 \otimes e_i(b_3)$$

EX $\mathbb{F} = A_r$ type $GL(rn)$ $rn = n$

Recall $B = B^{(1)}$ is semi-normal with crystal graph



For $1 \leq i \leq r$

$$\varphi_i \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array} \right) = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j \neq i \end{cases} \quad 1 \leq j \leq n$$

$$\varepsilon_i \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array} \right) = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{if } j \neq i \end{cases} \quad 1 \leq j \leq n$$

For $k \geq 1$ consider crystal

$$\underbrace{B \otimes B \otimes \dots \otimes B}_k$$

For a vertex

$$x = b_1 \otimes b_2 \otimes \dots \otimes b_k$$

and $1 \leq i \leq r$ find

$$\varphi_i(x), \quad \varepsilon_i(x), \quad f_i(x), \quad e_i(x)$$

For $1 \leq j \leq k$, under by put

" \supset " if $b_j = \boxed{i}$

" \subset " if $b_j = \boxed{i+1}$

After cancellation obtain

$\supset \supset \dots \supset \supset \quad \subset \subset \dots \subset \subset$

*

then

$\underbrace{\supset \supset \dots \supset \supset}_{\varphi_i(x)} \quad \underbrace{\subset \subset \dots \subset \subset}_{\varepsilon_i(x)}$

To obtain $f_i(x)$, replace \boxed{i} by $\boxed{i+1}$ above

the rightmost \supset in *

To obtain $\varepsilon_i(x)$ replace $\boxed{i+1}$ by \boxed{i} above

the leftmost \subset in *

To illustrate, take $r=2$, $k=10$

$$x = 2\ 1\ 3\ 3\ | 2\ 2\ 2\ 1\ 1$$

suppress notation
⊗, □

For $i=1, 2$ find

$$\varphi_i(x), \varepsilon_i(x), f_i(x), e_i(x)$$

$i=1$

$$\begin{array}{cccccccc} 2 & 1 & 3 & 3 & 1 & 2 & 2 & 2 & 1 & 1 \\ \langle & & & & & \rangle & \langle & \langle & \rangle & \rangle \\ \uparrow & & & & \uparrow & & & & & \end{array}$$

key
1 2
> <

$$\varphi_1(x) = 1, \quad \varepsilon_1(x) = 1$$

$$f_1(x) = 2\ 1\ 3\ 3\ 2\ 2\ 2\ 1\ 1$$

$$e_1(x) = 2\ 1\ 3\ 3\ 1\ 1\ 2\ 2\ 1\ 1$$

$i=2$

$$\begin{array}{cccccccc} 2 & 1 & 3 & 3 & 1 & 2 & 2 & 2 & 1 & 1 \\ \rangle & & & & \langle & \langle & \rangle & \rangle & & \\ & & & & & & & \uparrow & & \end{array}$$

key
2 3
> <

$$\varphi_2(x) = 2, \quad \varepsilon_2(x) = 0$$

$$f_2(x) = 2\ 1\ 3\ 3\ 1\ 2\ 2\ 3\ 1\ 1$$

$$e_2(x) = \phi$$



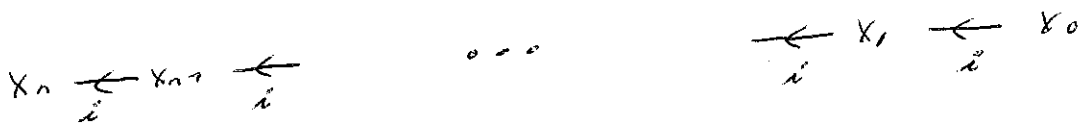
Given semi normal crystal B with
root data $\Phi, \Lambda, \Sigma = \{\alpha_i\}_{i \in I}$

For $i \in I$, an α_i -root string for B is a
sequence $\{x_j\}_{j=0}^{\hat{n}}$ of vertices in B st

$$f_i(x_j) = x_{j+1} \quad (0 \leq j < n), \quad f_i(x_n) = \phi.$$

$$e_i(x_j) = x_{j-1} \quad (1 \leq j \leq n), \quad e_i(x_0) = \phi.$$

So in the crystal graph,



B is disjoint union of α_i -root strings.

Describe functions f_i, e_i, wt on B

Write $\lambda = wt(x_0)$

We have

$$\begin{array}{ccccccc}
 & X_n & \leftarrow & X_{n-1} & \leftarrow & \dots & \leftarrow & X_1 & \leftarrow & X_0 \\
 \varphi_i & 0 & & 1 & & \dots & & n-1 & & n \\
 \varepsilon_i & n & & n-1 & & \dots & & 1 & & 0 \\
 wt & \lambda - n\alpha_i & & \lambda - (n-1)\alpha_i & & \dots & & \lambda - \alpha_i & & \lambda
 \end{array}$$

obs

$$\begin{aligned}
 n &= \varphi_i(x_0) - \varepsilon_i(x_0) \\
 &= \langle wt(x_0), \alpha_i^\vee \rangle \\
 &= \langle \lambda, \alpha_i^\vee \rangle
 \end{aligned}$$

Recall reflection

$$\begin{array}{ccc}
 \Lambda & \longrightarrow & \Lambda \\
 \mu & \longrightarrow & \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i
 \end{array}$$

So

$$\alpha_i(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha_i^\vee \rangle}_n \alpha_i$$

For $0 \leq j \leq n$,

$$\underbrace{\alpha_i(\underbrace{wt(x_{j+1})}_{\lambda - j\alpha_i})}_{\lambda - n\alpha_i + j\alpha_i} = \underbrace{wt(x_{j+1})}_{\lambda - (j-1)\alpha_i}$$

We now use s_i to define a bijection

$$\sigma_i : B \rightarrow B$$

The map σ_i permutes the vertices in each d_i -root string.

For the d_i -root string $\{x_j\}_{j=0}^{\hat{n}}$,

$$\sigma_i(x_j) = x_{n-j} \quad \text{as } j \in n$$

So
$$\sigma_i^2 = 1$$

LEM For $i \in I$ and $x \in B$,

$$\text{wt}(\sigma_i(x)) = s_i(\text{wt}(x))$$

pf x is contained in d_i -root string $x_0, \bar{x}_1, \dots, x_n$
write $x = x_j$

$$\text{wt}(\underbrace{\sigma_i(x_j)}_{x_{n-j}}) \stackrel{?}{=} \underbrace{s_i(\text{wt}(x_j))}_{\text{wt}(x_{n-j})}$$



LEM For $i \in I$ and $x \in B_i$

write

$$k = \langle wt(x), d_i^v \rangle_0$$

then

$$\sigma_i(x) = \begin{cases} f_i^k(x) & \text{if } k > 0 \\ x & \text{if } k = 0 \\ e_i^{-k}(x) & \text{if } k < 0 \end{cases}$$

pf x is contained in d_i -root string x_0, x_1, \dots, x_n

write $x = x_j$

$$k = \underbrace{\langle \overbrace{wt(x_j)}^{\lambda - j\alpha_i}, d_i^v \rangle}_{n-2j}$$

Case $k \geq 0$

$$\begin{array}{ccc} \sigma_i(x_j) & \stackrel{?}{=} & f_i^{n-2j}(x_j) \\ \parallel & & \parallel \\ x_{n-j} & \text{ok} & x_{j+n-2j} \end{array}$$

Case $k \leq 0$

$$\begin{array}{ccc} \sigma_i(x_j) & \stackrel{?}{=} & e_i^{2j-n}(x_j) \\ \parallel & & \parallel \\ x_{n-j} & \text{ok} & x_{j-(2j-n)} \end{array}$$

□