Lecture 10 Wednesday Sept 25

Signature Rule

Given seminormal crystals $B_1, B_2, \ldots, B_k$ with same root data.

Given $\underset{\underset{x}{\parallel}}{b_1 \otimes \cdots \otimes b_k} \in B_1 \otimes \cdots \otimes B_k$ and $i \in I$

Find $\varphi_i(x)$, $\varepsilon_i(x)$, $f_i(x)$, $e_i(x)$

Attach to x a finite sequence involving 2 kinds of symbols

open parenthesis : $($

closed parenthesis : $)$

For $1 \le j \le k$, b_j contributes

$$\underbrace{)) \cdots))}_{\varphi_i(b_j)}\ \underbrace{((\cdots ((}_{\varepsilon_i(b_j)}$$

Cancel all matched pairs of adjacent parentheses $()$ and iterate to obtain a sequence

$$)) \cdots))\ ((\cdots ((\qquad \ast$$

then

$$\underbrace{)) \cdots))}_{\varphi_i(x)}\ \underbrace{((\cdots ((}_{\varepsilon_i(x)}$$

ex $\qquad k = 3$

$$x = b_1 \otimes b_2 \otimes b_3$$

b	b_1	b_2	b_3
$\varphi_i(b)$	1	3	2
$\varepsilon_i(b)$	4	1	3

$$b_1 \quad \otimes \quad b_2 \quad \otimes \quad b_3$$

$$> \; <<<< \qquad >>> \; < \qquad >> \; <<<$$

$$> \; < \; <<< \;\; >>> \;\; < \;\; >> \; <<<$$

$$\underbrace{>}_{1} \; \underbrace{<<<}_{3}$$

$\varphi_i(x) = 1, \qquad \varepsilon_i(x) = 3$

Referring to $x = b_1 \otimes \cdots \otimes b_k$

find $f_i(x)$, $e_i(x)$.

We have

$$f_i(x) = b_1 \otimes \cdots \otimes f_i(b_t) \otimes \cdots \otimes b_k$$

$$e_i(x) = b_1 \otimes \cdots \otimes e_i(b_s) \otimes \cdots \otimes b_k$$

Find t, s:

b_t is attached to rightmost surviving $>$ in $*$:

$$\underset{\uparrow}{>> \cdots >>} << \cdots <<$$

If no such $>$ then $f_i(x) = \phi$

b_s is attached to leftmost surviving $<$ in $*$:

$$>> \cdots >> \underset{\uparrow}{<<} \cdots <<$$

If no such $<$ then $e_i(x) = \phi$

Ex $k = 3$, revisited

$$x = \overset{b_1 \quad\quad \otimes \quad\quad b_2 \quad\quad \otimes \quad\quad b_3}{> < < < < > > > < > > < < <}$$

rightmost surviving $>$ attached to b_1

leftmost surviving $<$ attached to b_3

$f_i(x) = f_i(b_1) \otimes b_2 \otimes b_3$

$e_i(x) = b_1 \otimes b_2 \otimes e_i(b_3)$

Ex $\Phi = A_r$ type $GL(m)$ $m = n$

Recall $\underset{B}{\overset{=}{B}}_{(1)}$ is seminormal with crystal graph

$$\boxed{n} \xleftarrow{n-1} \boxed{n-1} \xleftarrow{n-2} \cdots \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}$$

For $1 \le i \le r$

$$\varphi_i\left(\boxed{j}\right) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \ne i \end{cases} \qquad 1 \le j \le n$$

$$\varepsilon_i\left(\boxed{j}\right) = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{if } j \ne i+1 \end{cases} \qquad 1 \le j \le n$$

For $k \ge 1$ consider crystal

$$\underbrace{B \otimes B \otimes \cdots \otimes B}_{k}$$

For a vertex

$$x = b_1 \otimes b_2 \otimes \cdots \otimes b_k$$

and $1 \le i \le r$ find

$$\varphi_i(x), \quad \varepsilon_i(x), \quad f_i(x), \quad e_i(x)$$

For $1 \le j \le k$, under b_j put

"$)$" if $b_j = \boxed{i}$

"$($" if $b_j = \boxed{i+1}$

After cancellation obtain

$$)) \cdots)) \; ((\cdots ((\qquad *$$

Then

$$\underbrace{)) \cdots))}_{\varphi_i(x)} \; \underbrace{((\cdots ((}_{\varepsilon_i(x)}$$

To obtain $f_i(x)$, replace $\boxed{i}$ by $\boxed{i+1}$ above the rightmost $)$ in $*$

To obtain $e_i(x)$ replace $\boxed{i+1}$ by $\boxed{i}$ above the leftmost $($ in $*$

To illustrate, take $r = 2$, $k = 10$

suppress notation $\otimes$, $\square$

$$x = 2\ 1\ 3\ 3\ 1\ 2\ 2\ 2\ 1\ 1$$

For $i = 1, 2$ find

$\varphi_i(x)$, $\varepsilon_i(x)$, $f_i(x)$, $e_i(x)$

$i = 1$

$$\begin{array}{cccccccccc} 2 & 1 & 3 & 3 & 1 & 2 & 2 & 2 & 1 & 1 \\ (&) & & &) & (& (& (&) &) \\ & & & & \uparrow & \Uparrow & & & & \end{array}$$

key

$$\begin{array}{cc} 1 & 2 \\ ) & (\end{array}$$

$\varphi_1(x) = 1$, $\varepsilon_1(x) = 1$

$f_1(x) = 2\ 1\ 3\ 3\ 2\ 2\ 2\ 2\ 1\ 1$

$e_1(x) = 2\ 1\ 3\ 3\ 1\ 1\ 2\ 2\ 1\ 1$

$i = 2$

$$\begin{array}{cccccccccc} 2 & 1 & 3 & 3 & 1 & 2 & 2 & 2 & 1 & 1 \\ ) & & (& (& &) &) &) & & \\ & & & & & & & \uparrow & & \end{array}$$

key

$$\begin{array}{cc} 2 & 3 \\ ) & (\end{array}$$

$\varphi_2(x) = 2$, $\varepsilon_2(x) = 0$

$f_2(x) = 2\ 1\ 3\ 3\ 1\ 2\ 2\ 3\ 1\ 1$

$e_2(x) = \phi$

$\square$

Given semi normal crystal $\mathcal{B}$ with root data Φ, Λ, $\Sigma = \{\alpha_i\}_{i \in I}$

For $i \in I$, an <u>α_i - root string</u> for $\mathcal{B}$ is a sequence $\{x_j\}_{j=0}^{n}$ of vertices in $\mathcal{B}$ st

$$f_i(x_j) = x_{j+1} \quad (0 \le j \le n-1), \qquad f_i(x_n) = \phi.$$

$$e_i(x_j) = x_{j-1} \quad (1 \le j \le n), \qquad e_i(x_0) = \phi.$$

So in the crystal graph,

$$x_n \xleftarrow[i]{} x_{n-1} \xleftarrow[i]{} \cdots \xleftarrow[i]{} x_1 \xleftarrow[i]{} x_0 \qquad \ast$$

$\mathcal{B}$ is disjoint union of α_i - root strings.

Describe functions φ_i, ε_i, wt on $\ast$

Write $\lambda = wt(x_0)$

We have

$$\begin{array}{ccccccc} & x_n \leftarrow & x_{n-1} \leftarrow & \cdots & \leftarrow & x_1 \leftarrow & x_0 \\ \varphi_i & 0 & 1 & \cdots & & n-1 & n \\ \varepsilon_i & n & n-1 & \cdots & & 1 & 0 \\ wt & \lambda - n\alpha_i & \lambda-(n-1)\alpha_i & \cdots & & \lambda-\alpha_i & \lambda \end{array}$$

obs

$$\begin{aligned} n &= \varphi_i(x_0) - \varepsilon_i(x_0) \\ &= \langle wt(x_0), \alpha_i^\vee \rangle \\ &= \langle \lambda, \alpha_i^\vee \rangle \end{aligned}$$

Recall reflection

$$\begin{aligned} s_i : \quad \Lambda &\longrightarrow \Lambda \\ \mu &\longrightarrow \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i \end{aligned}$$

So

$$s_i(\lambda) = \lambda - \underbrace{\langle \lambda, \alpha_i^\vee \rangle}_{= n} \alpha_i$$

For $0 \le j \le n$,

$$\underbrace{s_i\big(\underbrace{wt(x_j)}_{= \lambda - j\alpha_i}\big)}_{\lambda - n\alpha_i + j\alpha_i} = \underbrace{wt(x_{n-j})}_{= \lambda - (n-j)\alpha_i}$$

We now use s_i to define a bijection

$$\sigma_i : \mathcal{B} \longrightarrow \mathcal{B}$$

The map σ_i permutes the vertices in each α_i-root string.

For the α_i-root string $\{x_j\}_{j=0}^{n}$,

$$\sigma_i(x_j) = x_{n-j} \qquad 0 \le j \le n$$

So $$\sigma_i^2 = 1$$

LEM For $i \in I$ and $x \in \mathcal{B}$,

$$wt(\sigma_i(x)) = s_i(wt(x))$$

pf x is contained in α_i-root string $x_0, x_1, \dots, x_n$.
write $x = x_j$

$$wt(\underbrace{\sigma_i(x_j)}_{= x_{n-j}}) \overset{?}{=} \underbrace{s_i(wt(x_j))}_{= wt(x_{n-j})}$$

$\square$

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LEM Fn $i \in I$ and $x \in B_1$

write

$$k = \langle wt(x), \alpha_i^\vee \rangle.$$

Then

$$\sigma_i(x) = \begin{cases} f_i^k(x) & \text{if } k > 0 \\ x & \text{if } k = 0 \\ e_i^{-k}(x) & \text{if } k < 0 \end{cases}$$

pf x is contained in α_i-root string $x_0, x_1, \ldots, x_n$

write $x = x_j$

$$k = \underbrace{\langle \overbrace{wt(x_j)}^{\lambda - j\alpha_i}, \alpha_i^\vee \rangle}_{n-2j}$$

<u>Case $k \geq 0$</u>

$$\underset{\substack{\| \\ x_{n-j}}}{\sigma_i(x_j)} \overset{?}{=} \underset{\substack{\| \\ x_{j+n-2j}}}{f_i^{n-2j}(x_j)}$$

ok

<u>Case $k \leq 0$</u>

$$\underset{\substack{\| \\ x_{n-j}}}{\sigma_i(x_j)} \overset{?}{=} \underset{\substack{\| \\ x_{j-(2j-n)}}}{e_i^{2j-n}(x_j)}$$

ok

$\square$