WISCONSIN
UNIVERSITY OF WISCONSIN–MADISON

Math 846: Crystal Bases in Algebraic Combinatorics
Lecture 001, MWF 1:20–2:10, Van Vleck B113
Syllabus for Semester I, 2019/2020

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Text: Crystal Bases, by Daniel Bump and Anne Schilling. World Scientific, 2017.
ISBN: 978 981 473 3441

Prerequisites: Good understanding of linear algebra.

Course Content: A crystal base is a purely combinatorial object that is used to describe representations of Lie algebras and quantum groups. In this introductory course, we will develop the theory of crystal bases from first principles, and see how they get used in representation theory. Along the way we will encounter topics such as: root systems, Kashiwara crystals, Young tableaux and their crystals, Stembridge crystals, insertion algorithms, bicrystals and the Littlewood-Richardson rule, crystals for Stanley symmetric functions, and Gelfand-Tsetlin patterns.

The lectures will be self contained and no prior knowledge of the subject is assumed. I will follow the text more or less. This course is suitable for first year graduate students. It is recommended for anyone interested in algebraic combinatorics, representation theory, Lie theory, quantum groups, and statistical mechanical models.

Course Credits: 3. Each week there will be three 50 minute lectures.

Evaluation: There are no exams. Near the end of the semester each non-dissertator student is expected to give one lecture, on a topic either from the text or a related topic of your choice. As the time approaches I will organize the speaking schedule and suggest topics.

Course goals/Learning outcomes: Master the material presented in lecture. For this I recommend the following study strategy. After each lecture do the following: for each stated definition write out numerous examples and non examples. For each stated result, write your own proof starting from first principles and without looking at your notes. It is not important if your proof matches mine or not. Done properly this strategy is easy to carry out, since every result in the course builds naturally on what came before.

Lecture 1 Wed Sept 4 9/4/19
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Fall 2019

MATH 846

Crystal Bases in Algebraic Combinatorics

Paul Terwilliger

- See syllabus
- Main topic of crystals uses facts about root systems. So we start by discussing root systems

Root systems

A <u>Euclidean space</u> is a finite dim'l vector space V over $\mathbb{R}$, together with a positive definite sym bilinear form $\langle , \rangle : V \times V \to \mathbb{R}$

So $\langle u,v \rangle = \langle v,u \rangle$ $\quad u, v, u' \in V$, $a \in \mathbb{R}$

$\langle au, v \rangle = a \langle u,v \rangle$

$\langle u+u', v \rangle = \langle u,v \rangle + \langle u', v \rangle$

$\langle u,u \rangle \geq 0$

$\langle u,u \rangle = 0$ iff $u = 0$

Assume $V, \langle , \rangle$ is a Euclidean space dim n

V has a basis $\{e_i\}_{i=1}^n$ st $\quad$ "orthonormal" basis

$\langle e_i, e_j \rangle = \delta_{ij} \quad 1 \leq i, j \leq n$

To be concrete, often view

$$V = \mathbb{R}^n \qquad \text{(column vectors)}$$

For

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\langle u, v \rangle = u^t v = \sum_{i=1}^{n} u_i v_i$$

Take

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i \qquad 1 \le i \le n$$

Call $\{e_i\}_{i=1}^n$ the <u>standard basis</u> for V

— ∘ —

From now on fix a nonzero Euclidean space V

Define

$$O(V) = \left\{ \sigma \in GL(V) \;\middle|\; \begin{array}{l} \langle \sigma u, \sigma v\rangle = \langle u, v\rangle \\ \forall u, v \in V \end{array} \right\}$$

"orthogonal group"

For a subset $T \subseteq V$ define

$$T^\perp = \{ v \in V \mid \langle v, t\rangle = 0 \;\; \forall t \in T \}$$

"orthogonal complement of T"

For $0 \neq \alpha \in V$ we have

$$V = \mathbb{R}\alpha + \alpha^\perp \qquad \text{(dir sum)}$$

Define

$$r_\alpha : \begin{array}{ccc} V & \longrightarrow & V \\ v & \longrightarrow & v - \dfrac{2\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \end{array}$$

Obs

subspace U	$\mathbb{R}\alpha$	$\alpha^\perp$
r_α action on U	$-I$	I

"reflection"

So $r_\alpha \in O(V)$, $\quad r_\alpha^2 = I$

Def 1 A <u>root</u> <u>system</u> in V is a nonempty finite set Φ of non 0 vectors in V, such that

(1) $r_\alpha(\Phi) = \Phi$ $\alpha \in \Phi$

(2) $\dfrac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ $\alpha, \beta \in \Phi$

(3) For $\alpha \in \Phi$ and $a \in \mathbb{R}$,
$a\alpha \in \Phi$ implies $a = \mp 1$.

An element of Φ is called a <u>root</u>

<u>Caution</u> We do not assume Φ spans V

For a root system Φ in V
For a subspace $W \leq V$ such that $\Phi \cap W \neq \phi$
then $\Phi \cap W$ is a root system in W.

Ex For $\dim V = 2$ find all the root systems Φ in V

Case Φ does not span V

$$\Phi = \{ \alpha, -\alpha \}$$

Case Φ spans V

$\exists$ linearly indep $\alpha, \beta \in \Phi$
choose α, β so that their angle is maximal

obs $\langle \alpha, \beta \rangle \leq 0$
otherwise we can increase the angle by replacing $\alpha \to -\alpha$

We have

$$0 \geq \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$$

$$0 \geq \frac{2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$$

So

$$\frac{4\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle} \qquad \ast$$

is non neg integer.

Also the matrix of inner products

$$\begin{pmatrix} \langle\alpha,\alpha\rangle & \langle\alpha,\beta\rangle \\ \langle\beta,\alpha\rangle & \langle\beta,\beta\rangle \end{pmatrix}$$

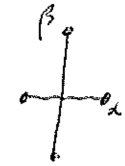
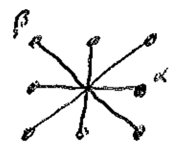
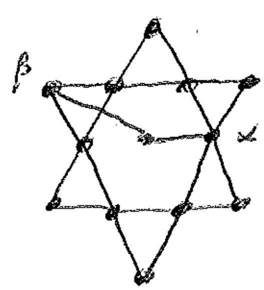
is pos def, so

$$\frac{\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle} < 1$$

So $\ast \in \{0,1,2,3\}$

Interchanging α,β if nec, WLOG

$$\langle\alpha,\alpha\rangle \leq \langle\beta,\beta\rangle$$

$*$	$\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$	$\frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$	Φ	$\lvert\Phi\rvert$
0	0	0		4
1	-1	-1		6
2	-2	-1		8
3	-3	-1		12

Given a root system Φ in V

For a root $\alpha \in \Phi$

Define

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

"coroot for α"

The set

$$\Phi^\vee = \{ \alpha^\vee \mid \alpha \in \Phi \}$$

is a root system in V

The root system Φ is <u>reducible</u> whenever

$$\Phi = \Phi_1 \cup \Phi_2$$

$$\langle \Phi_1, \Phi_2 \rangle = 0$$

Φ_1, Φ_2 root systems

Φ is <u>simple</u> or <u>irreducible</u> whenever Φ is not reducible

Φ is <u>simply laced</u> whenever

$\langle \alpha, \alpha \rangle$ is indep of α for $\alpha \in \Phi$

A <u>weight lattice</u> of Φ is subgroup Λ of the abelian group $V, +$ such that

(1) Λ spans V

(2) $\Phi \subseteq \Lambda$

(3) $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ $\quad \lambda \in \Lambda, \alpha \in \Phi$

Each $\lambda \in \Lambda$ called a <u>weight</u>

In the following examples assume $\dim V = 2$

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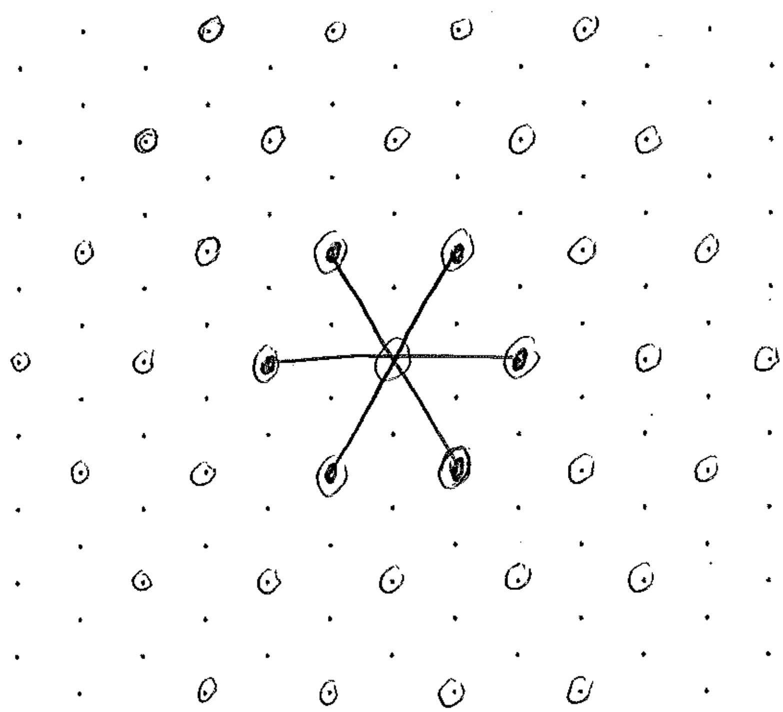
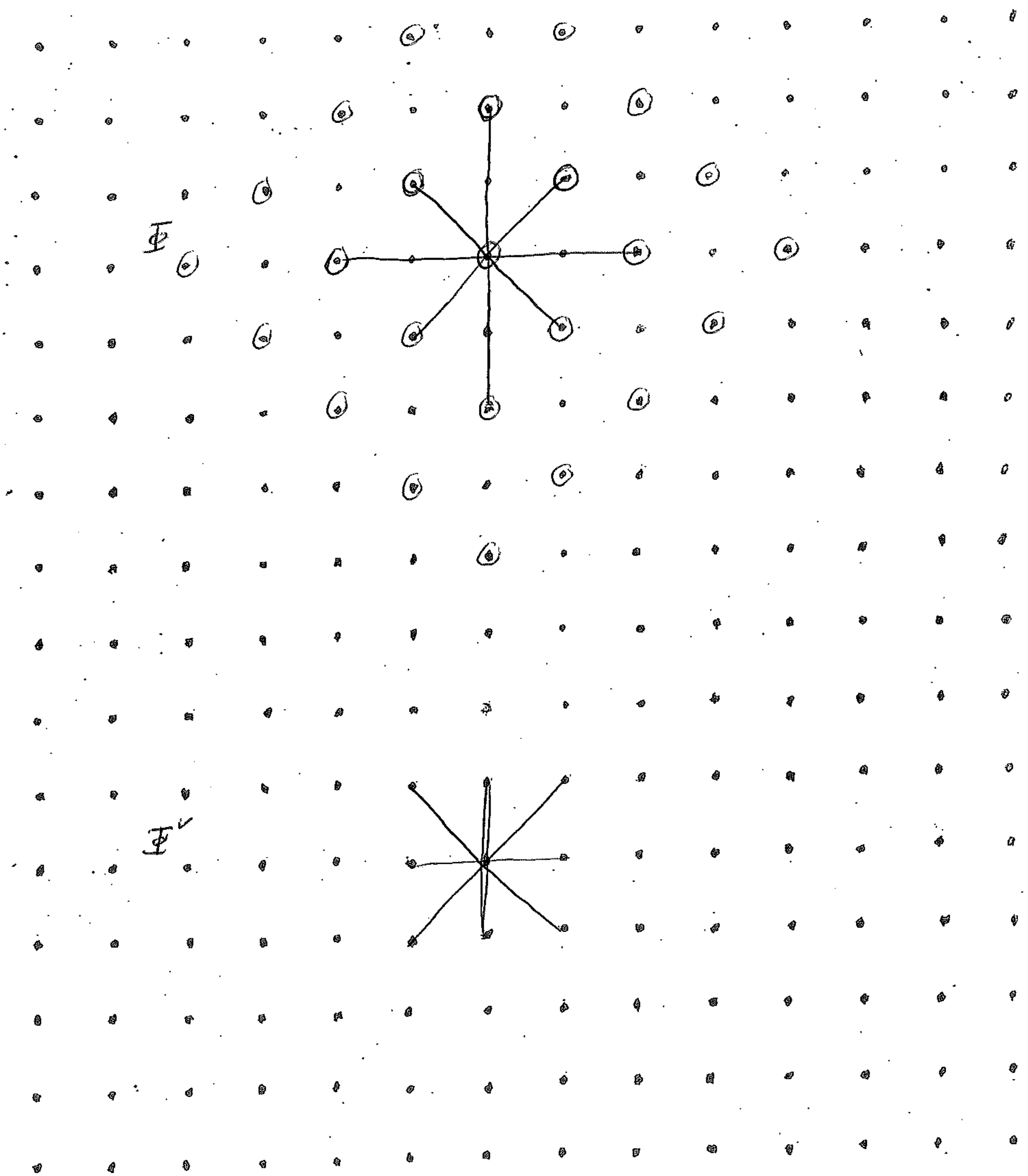


$\Phi = \Phi^{\vee}$

$\Lambda =$ fine dots. (Λ_{sc})

or $\Lambda =$ circled dots. (Λ_{root})

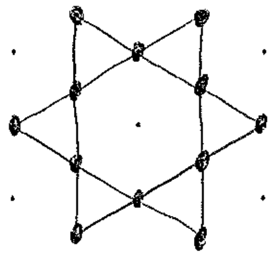
$\Lambda =$ fine dots (Λ_{sc}) or $\Lambda =$ circled dots (Λ_{root})



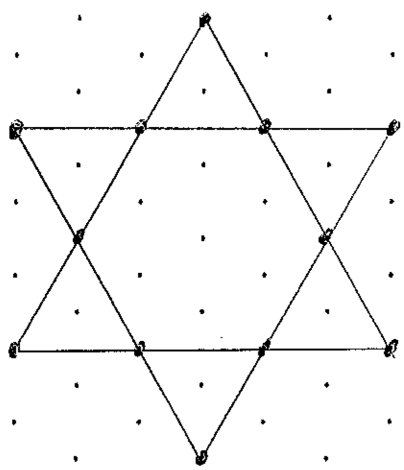
Λ = fine dots $= \mathbb{Z}\Phi$

$(\Lambda_{sc} = \Lambda_{root})$

Φ



$\Phi^\vee$



A root system Φ in V is <u>semi simple</u>

whenever Φ spans V

Define

$$\Lambda_{root} = \mathbb{Z}\,\Phi$$

= set of linear combinations of roots with integer coefs.

"root lattice"

For a weight lattice Λ of Φ,

$$\Lambda_{root} \leq \Lambda$$

Λ_{root} is a weight Lattice if Φ is s.s

In this case Λ_{root} has <u>adjoint type</u>

LEM For a root system Φ in V and a wt Lattice Λ for Φ TFAE

(i) Φ is semi simple

(ii) $\mathbb{Z}\Phi$ has finite index in Λ

pf (i) $\to$ (ii) Φ spans V

V has basis of roots

$\alpha_1, \alpha_2, \dots, \alpha_n$

Fr $\lambda \in \Lambda$

$$\underbrace{\langle \alpha_i^\vee, \lambda \rangle}_{= b_i} \in \mathbb{Z} \qquad 1 \le i \le n$$

Write $\lambda = \sum_{j=1}^{n} x_j \alpha_j$ $\qquad x_j \in \mathbb{R}$

Fr $1 \le i \le n$

$$b_i = \langle \alpha_i^\vee, \lambda \rangle = \sum_{j=1}^{n} \underbrace{\langle \alpha_i^\vee, \alpha_j \rangle}_{\in \mathbb{Z}} x_j$$

$$\underbrace{\Big(\langle \alpha_i^\vee, \alpha_j \rangle \Big)_{1 \le i, j \le n}}_{= A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$x_i = \frac{\text{integer}}{\det A} \qquad 1 \le i \le n$$

$$(\det A)\,\lambda \in \mathbb{Z}\Phi$$

We have

$$\mathbb{Z}\Phi \subseteq \Lambda \subseteq \frac{1}{\det A}\,\mathbb{Z}\Phi$$

finite index $(\det A)^n$

So $\mathbb{Z}\Phi$ has finite index in Λ

(ii) $\to$ (i) Suppose Φ not ss

So Φ does not span V

Since Λ spans V

$\exists$ $\lambda \in \Lambda$ st

$$\lambda \notin \mathbb{R}\Phi$$

Consider cosets of $\mathbb{Z}\Phi$ in Λ

the elements

$$\lambda, 2\lambda, 3\lambda, \dots$$

are in distinct cosets. Indeed for $1 \leq i < j < \infty$

if $i\lambda, j\lambda$ in same coset then

$$j\lambda - i\lambda \in \mathbb{Z}\Phi$$

forcing $\lambda \in \mathbb{R}\Phi$ cnt.

So $\mathbb{Z}\Phi$ has ∞ many cosets in Λ cnt.

$\square$