Lec 9 Monday Sept 26

Given an algebra A with mult

$$m: \begin{array}{ccc} A \otimes A & \to & A \\ a \otimes b & \to & ab \end{array}$$

Recall the opposite algebra A^{op} has mult

$$m^{op}: \begin{array}{ccc} A \otimes A & \to & A \\ a \otimes b & \to & ba \end{array}$$

Given a coalg C with comult

$$\Delta: \begin{array}{ccc} C & \to & C \otimes C \\ c & \to & \sum_{(c)} c_1 \otimes c_2 \end{array}$$

the opposite coalg C^{op} has comult

$$\Delta^{op}: \begin{array}{ccc} C & \to & C \otimes C \\ c & \to & \sum_{(c)} c_2 \otimes c_1 \end{array}$$

Given a bialgebra H with data m, Δ.

Get four bialgebras with data

m, Δ	m^{op}, Δ
m, Δ^{op}	m^{op}, Δ^{op}

"relatives"

Next goal:

Suppose H is Hopf alg with antipode S.

Are H's relatives also Hopf algebras, and if so, what are their antipodes?

Case m^{op}, Δ

Antipode $\tilde{S}$ must satisfy:

$\forall a \in H$,

$$\underbrace{\sum_{(a)} a_1 \tilde{S}(a_2)}_{\substack{\uparrow \\ m^{op}}} = \varepsilon(a) 1_H = \underbrace{\sum_{(a)} \tilde{S}(a_1) a_2}_{\substack{\uparrow \\ m^{op}}}$$

$$\sum_{(a)} a_1 \tilde{S}(a_2) = \sum_{(a)} \tilde{S}(a_2) a_1$$

$$\sum_{(a)} \tilde{S}(a_1) a_2 = \sum_{(a)} a_2 \tilde{S}(a_1)$$

By prev LEM,

$\tilde{S}$ exists iff S^{-1} exists, and in this case

$$\tilde{S} = S^{-1}$$

Case m, Δ^{op}

Antipode $S^\vee$ must satisfy:

$\forall a \in H,$

$$\sum_{(a)} a_1 S^\vee(a_2) = \varepsilon(a) 1_H = \sum_{(a)} S^\vee(a_1) a_2 \qquad (\bullet)$$

Here

$$\sum_{(a)} a_1 \otimes a_2 = \Delta^{op}(a) = \sum_{(a)} a_2 \otimes a_1$$
(wrt Δ)

In terms of Δ, $(\bullet)$ becomes

$$\sum_{(a)} a_2 S^\vee(a_1) = \varepsilon(a) 1_H = \sum_{(a)} S^\vee(a_2) a_1$$

By prev LEM,

$S^\vee$ exists iff S^{-1} exists, and in this case

$$S^\vee = S^{-1}$$

Case $m^{\sigma}, \Delta^{\sigma}$

this bialgebra has antipode S (ex) $\square$

Prop Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n \qquad (\bullet)$$

Recall H has an antipode S

Then S^{-1} exists.

pf For the bialg H, m^{op}, Δ

$(\bullet)$ is still a connected grading

So its antipode exists.

But this antipode is S^{-1} by previous comments. $\square$

Next goal: Given coalgebra C

Define a subcoalg of C

Aside on tensor products

Given k-module V

Given k-submodule $U \subseteq V$

incl map

$$i : U \to V$$

is injective k-mod hom.

Consider the k mod hom

$$i \otimes i : \begin{array}{l} U \otimes U \to V \otimes V \\ a \otimes b \to a \otimes b \end{array} \qquad (*)$$

$(*)$ might not be injective, as the next example shows.

Ex Given

$\mathbb{F}$ = a field

x = indeterminate

$K = \mathbb{F}[x]$ polys in x

Obs K is com. ring with 1.

let V = vector space over $\mathbb{F}$ with dimension 2

Pick a basis e, f for V

V becomes a K-module with x-action

$xe = 0, \quad xf = e$

let U = subspace of V with basis e

Obs U is K-submodule of V

Consider incl map

$$i: \begin{array}{ccc} U & \to & V \\ e & \to & e \end{array}$$

Describe $U \otimes U$, $V \otimes V$ $\quad \otimes = \otimes_K$

<u>$U \otimes U$</u> As a vector space over $\mathbb{F}$,

$U \otimes U$ has basis $e \otimes e$

Moreover

$$x(e \otimes e) = \underset{\overset{\shortparallel}{0}}{(xe)} \otimes e = 0$$

<u>$V \otimes V$</u> As a vector space over $\mathbb{F}$

$V \otimes V$ has a basis

$$e \otimes f = f \otimes e, \qquad f \otimes f$$

Moreover

$$x(e \otimes f) = \underset{\overset{\shortparallel}{0}}{(xe)} \otimes f = 0$$

$$x(f \otimes f) = (xf) \otimes f = e \otimes f$$

Obs
$$e \otimes e = (xf) \otimes e$$
$$= f \otimes \underset{\overset{\shortparallel}{0}}{(xe)}$$
$$= 0$$

Now $i \otimes i : U \otimes U \to V \otimes V$ sends

$$\underset{\overset{\#}{0}}{e \otimes e} \longrightarrow \underset{\overset{\shortparallel}{0}}{e \otimes e}$$

$\square$

Ex Given a k-module V,

Given a k-submodule $U \subseteq V$

Consider incl map

$$i : U \to V$$

Assume $\exists$ k-module W of V st

$$V = U + W \quad (ds)$$

"W is k-module complement of U in V"

then the k-module hom

$$i \otimes i : U \otimes U \to V \otimes V$$

is injective.

pf The k-module iso $V \simeq U \oplus W$

induces k-module isomorphisms

$$V \otimes V \simeq (U \oplus W) \otimes (U \oplus W)$$

$$\simeq (U \otimes U) \oplus (U \otimes W) \oplus (W \otimes U) \oplus (W \otimes W)$$

Result follows. $\square$

Given k-coalgebra C

Define a subcoalg of C

Naive def: A subcoalg of C is a nonO k-submodule D of C st

$$\Delta_C(D) \subseteq i \otimes i (D \otimes D) \qquad i = \text{incl}$$

Using this defs lets turn D into a k-coalg st

$$i : D \to C$$

is a coalg morph.

First assume $i \otimes i$ is injective.
Via $i \otimes i$ identify $i \otimes i(D \otimes D)$ with $D \otimes D$

Define

$$\Delta_D : \begin{array}{ccc} D & \longrightarrow & D \otimes D \\ x & \rightarrow & \Delta_C(x) \end{array}$$

By cnstr $i : D \to C$ is coalg morphism.

Next assume $i \otimes i$ not injective.
Now the restriction of Δ_C to D does not induce a map $D \to D \otimes D$. D does not inherit a coalg str from C

So, naive def of subcoalg only works if $i \otimes i$ is injective.

Here is our official def of subcoalg

Def Given a k-coalg C

A <u>subcoalgebra of C</u> is a k-coalg D together with an injective coalg morphism

$$\theta : D \to C$$

[If we identify D with a k-submodule of C via θ, then $\theta : D \to C$ becomes the incl map]

— o —

The defs of subbialgebra, subHopf algebra are similar.

REV Given k-modules U, V, U', V'

Given surjective k-module homs

$$\varphi : U \to U', \qquad \phi : V \to V'$$

then the k-mod hom

$$\begin{aligned} \varphi \otimes \phi : \quad U \otimes V &\to U' \otimes V' \\ u \otimes v &\to \varphi(u) \otimes \phi(v) \end{aligned}$$

has kernel

$$\ker(\varphi \otimes \phi) = \ker(\varphi) \otimes V + U \otimes \ker(\phi) \qquad (*)$$

Moreover if k is a field, then $(*)$ still holds if the surjectivity assumption is dropped.

LEM Assume K is a field.

Given a K-coalg C and a K-module U

Given a K-module hom $f : C \to U$

Consider the composition

$$\theta : \quad \begin{array}{ccccccc} C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C & \xrightarrow{id \otimes f \otimes id} & C \otimes U \otimes C \\ c & \longmapsto & \sum_{(c)} c_1 \otimes c_2 & \longmapsto & \sum_{(c)} c_1 \otimes c_2 \otimes c_3 & \longmapsto & \sum_{(c)} c_1 \otimes f(c_2) \otimes c_3 \end{array}$$

Then $\underset{J}{\overset{\shortparallel}{\ker(\theta)}}$ is a subcoalg of C

pf Since K is field, suf to show

$$\Delta(J) \subseteq J \otimes J$$

Show both

$$\Delta(J) \subseteq J \otimes C, \quad \leftarrow$$

$$\Delta(J) \subseteq C \otimes J$$

The following diagrams commute: