Lec 8 Friday Sept 23

Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

We saw H has an antipode S.

Next general goal: present <u>Takeuchi's Formula</u> for S

LEM For any Hopf algebra H and any primitive $a \in H$,

$$S(a) = -a$$

pf Recall

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

$$\varepsilon(a) = 0$$

$$\varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2$$

So

$$0 = S(a) 1 + \underset{\overset{\shortparallel}{1}}{S(1)} a$$

So $S(a) = -a$ $\square$

Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Recall convolution algebra $\operatorname{End}(H)$ with mult $\star$

Recall some elements in $\operatorname{End}(H)$:

$$\mathbb{1}: \begin{array}{ccc} H & \longrightarrow & H \\ a & \longrightarrow & \varepsilon(a) 1_H \end{array}$$

identity in conv alg

$$id: \begin{array}{ccc} H & \longrightarrow & H \\ a & \longrightarrow & a \end{array}$$

identity in usual alg

Define

$$f = id - \mathbb{1}$$

So

$$f: \begin{array}{ccc} H & \longrightarrow & H \\ a & \longrightarrow & a - \varepsilon(a) 1_H \end{array}$$

Thm (Takeuchi) With above notation, the antipode S of H satisfies

$$S = \mathbb{1} - f + f \star f - f \star f \star f + \cdots \qquad (*)$$

(As we will see, for $a \in H$ the sum $*$ becomes finite when evaluated at a)

pf We have

$$id = \mathbb{1} + f$$

Recall S is inverse of id wrt $\star$

So

$$S = (\mathbb{1} + f)^{-1} \qquad \text{(in conv alg)}$$

$$= \mathbb{1} - f + \underset{\substack{\uparrow \\ f \star f}}{f^2} - \underset{\substack{\uparrow \\ f \star f \star f}}{f^3} + \cdots \qquad \text{(in conv alg)}$$

We now show this sum is well defined. To do this, we show it is effectively finite upon evaluation.

We show

$$f\,(H_0) = 0$$

$$f \star f \quad (H_0 + H_1) = 0$$

$$f \star f \star f \quad (H_0 + H_1 + H_2) = 0$$

$$\cdots$$

check $f\,(H_0) = 0$:

Since H is connected

$$H_0 = K\,1_H$$

For $a \in H_0$ write

$$a = \alpha\,1_H \qquad \alpha \in K$$

$$\begin{aligned} f(a) &= a - \varepsilon(a)\,1_H \\ &= \alpha\,1_H - \varepsilon(\alpha 1_H)\,1_H \end{aligned}$$

$$\varepsilon(\alpha 1_H) = \alpha\,\underbrace{\varepsilon(1_H)}_{=1}\,1_H$$

$$= 0 \checkmark$$

For any integers r, n st $1 \le r < n$

show

$$\underbrace{f \star f \star \cdots \star f}_{n} \,(H_r) = 0$$

Recall for $\varphi, \phi \in End(H)$

$$(\varphi \star \phi)(a) = \sum_{(a)} \varphi(a_1)\,\phi(a_2) \qquad a \in H$$

So for $a \in H_r$

$$\underbrace{f \star f \star \cdots \star f}_{n} \,(a) = \sum_{(a)} f(a_1)\, f(a_2) \cdots f(a_n)$$

Recall Δ respects the grading, ie

$$\Delta(H_r) \subseteq H_0 \otimes H_r + H_1 \otimes H_{r-1} + \cdots + H_r \otimes H_0$$

In the Sweedler sum

$$\sum_{(a)} a_1 \otimes a_2 \otimes \cdots \otimes a_n$$

WLOG each term

$$a_1 \otimes a_2 \otimes \cdots \otimes a_n \in H_{i_1} \otimes H_{i_2} \otimes \cdots \otimes H_{i_n}$$

9/23/16
6

for some nonnegative integers $i_1, i_2, \ldots, i_n$

with $i_1 + i_2 + \cdots + i_n = r$

Since $n > r$, at least one of $i_1, i_2, \ldots, i_n$

is 0

So $\exists\, j\ (1 \le j \le n)$ st $i_j = 0$

So $a_j \in H_0$

So $f(a_j) = 0$

So $f(a_1) f(a_2) \cdots f(a_n) = 0$

Now

$$\underbrace{f \star f \star \cdots \star f}_{n}\ (a) = \sum_{(a_1} \underbrace{f(a_1) f(a_2) \cdots f(a_n)}_{= 0}$$

$$= 0$$

□

Next goal:

Given Hopf algebras H, H'

with antipodes S, S'

Given a bialgebra morphism $\varphi : H \to H'$

↑

alg morph + coalg morph

Then show $\varphi S = S' \varphi$.

LEM Given Hopf algebra H with antipode S.

Given an algebra A and algebra morphism

$\varphi : H \to A$.

Then in the convolution algebra $Hom(H, A)$

$\varphi, \quad \varphi S$ "$\star$ - inverses"

are inverses.

pf Recall the identity

$$\mathbb{1} : \begin{array}{ccc} H & \longrightarrow & A \\ x & \longmapsto & \varepsilon(x)\, 1_A \end{array}$$

Show

$$\varphi S \star \varphi = \mathbb{1}$$

$\forall x \in H$ show

$$\sum_{(x)} \varphi S(x_1)\, \varphi(x_2) \overset{?}{=} \varepsilon(x)\, 1_A$$

$$\underbrace{\varphi\Big(\underbrace{\sum_{(x)} S(x_1)\, x_2}_{= \varepsilon(x)\, 1_H} \Big)}_{= \varepsilon(x)\, \varphi(1_H)}$$

$\varphi(1_H) = 1_A$ ✓

Similarly

$$\varphi \star \varphi S = \mathbb{1}.$$

Result follows. $\square$

LEM Given a Hopf algebra H with antipode S.

Given a coalgebra C and a coalg morphism

$\varphi: C \longrightarrow H$. Then in the convolution algebra

$Hom(C, H)$

$$\varphi, \quad S\varphi$$

are inverses.

pf Show

$$S\varphi \star \varphi = \mathbb{1}$$

Fn $x \in C$ show

$$\sum_{(x)} S\varphi(x_1)\,\varphi(x_2) \overset{?}{=} \varepsilon_C(x)\, 1_H$$

$\forall y \in H$

$$\sum_{(y)} S(y_1)\, y_2 = \varepsilon_H(y)\, 1_H \qquad (\bullet)$$

Take $y = \varphi(x)$.

Since φ is a coalg morphism,

$$\Delta_H(y) = \sum_{(x)} \varphi(x_1) \otimes \varphi(x_2)$$

$$\varepsilon_H(y) = \varepsilon_C(x)$$

So $(\bullet)$ becomes

$$\sum_{(x)} S\varphi(x_1)\varphi(x_2) = \varepsilon_C(x) 1_H$$

✓

$\square$

Prop Given Hopf algebras H, H'

with antipodes S, S'.

Given a bialgebra morphism $\varphi : H \to H'$

then $\varphi S = S' \varphi$

pf In the Convolution alg $\mathrm{Hom}(H, H')$

each of φS, $S'\varphi$ is the inverse of φ.

So $\varphi S = S' \varphi$. $\square$

Given Hopf alg H with antipode S

In convolution algebra $End(H)$,

S is $\star$-invertible with $\star$-inverse id_H

Is S invertible [in usual algebra $End(H)$] ?

Maybe.

We now give some nec/suf conditions for S to be invertible

LEM Given Hopf alg H with antipode S
Given $T \in End(H)$ TFAE

(i) $\forall a \in H$

$$\sum_{(a)} T(a_2)\, a_1 = \varepsilon(a)\, 1_H$$

(ii) $\forall a \in H$

$$\sum_{(a)} a_2\, T(a_1) = \varepsilon(a)\, 1_H$$

(iii) S, T are inverses

pf (ii) → (iii) Show

$$ST = id_H = TS$$

Show $ST = id_H$

$\forall a \in H$

$$\sum_{(a)} T(a_2)\, a_1 = \varepsilon(a)\, 1_H$$

Apply S:

$$\sum_{(a)} S(a_1)\, ST(a_2) = \varepsilon(a)\, 1_H$$

So $S \star ST = \mathbb{1}$

Recall $S \star id_H = \mathbb{1}$

So $ST = id_H$

show $TS = id_H$

For $a \in H$ write $b = S(a)$

We have

$$\sum_{(b)} T(b_2)\, b_1 = \varepsilon(b)\, 1_H \qquad (\bullet)$$

Recall

$$\sum_{(b)} b_1 \otimes b_2 = \sum_{(a)} S(a_2) \otimes S(a_1)$$

and

$$\varepsilon(b) = \varepsilon(a)$$

So $(\bullet)$ becomes

$$\sum_{(a)} TS(a_1)\, S(a_2) = \varepsilon(a)\, 1_H$$

So

$$TS \star S = \mathbb{1}$$

Recall

$$id_H \star S = \mathbb{1}$$

So

$$TS = id_H$$

(iii) → (i)

$\forall\, a \in H$

$$\sum_{(a)} S(a_1)\, a_2 = \varepsilon(a)\, 1_H$$

Apply $T = S^{-1}$:

$$\sum_{(a)} S^{-1}(a_2)\, a_1 = \varepsilon(a)\, 1_H$$ ✓

(ii) ⟺ (iii) Similar. $\square$