Lec 7 Wednesday Sept 21 9/21/16

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Given Hopf algebra H with antipode S

Recall $S(ab) = S(b)\,S(a)$ $a, b \in H$

$S(1_H) = 1_H$

obs $S^2 = S \circ S$ satisfies

$S^2(ab) = S^2(a)\,S^2(b)$

so S^2 is an alg morphism.

LEM With above notation, assume either

(i) H is commutative,

(ii) H is cocommutative.

Then $S^2 = id$

pf (i) $\forall a \in H$

$$\varepsilon(a)\,1_H = \sum_{(a)} a_1 S(a_2) = \sum_{(a)} S(a_1)\,a_2$$

Apply S:

$$\varepsilon(a)\, S(1_H) = \sum_{(a)} S^2(a_2)\, S(a_1)$$

$$= \sum_{(a)} S(a_1)\, S^2(a_2) \qquad \text{by com}$$

So

$$0 = \sum_{(a)} S(a_1)\left(S^2(a_2) - a_2\right)$$

$$= \left(S \star (S^2 - id)\right)(a)$$

So

$$0 = S \star (S^2 - id)$$

But S is invertible wrt $\star$ so

$$0 = S^2 - id$$

So

$$S^2 = id$$

(ii) (Similar to pf of (i)) $\forall a \in H$

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2 = \sum_{(a)} a_2 \otimes a_1$$

So

$$\varepsilon(a)\, 1_H = \sum_{(a)} a_1 S(a_2) = \sum_{(a)} a_2 S(a_1)$$

Apply S:

$$\varepsilon(a) S(1_H) = \sum_{(a)} S^2(a_1) S(a_2)$$

So

$$0 = \sum_{(a)} \left(S^2(a_1) - a_1 \right) S(a_2)$$

$$= \left(\left(S^2 - id \right) \star S \right)(a)$$

So

$$0 = (S^2 - id) \star S$$

So

$$0 = S^2 - id$$

So

$$S^2 = id$$

$\square$

REV Given a k-algebra A.

The k-module A supports a k-algebra A^{op} with mult

$$ab \ (\text{in } A^{op}) = ba \ (\text{in } A)$$

For $f \in End(A)$ TFAE:

(i) $f(ab) = f(b)f(a) \quad \forall a,b \in A$ and

$$f(1_A) = 1_A$$

(ii) $f : A \to A^{op}$ is a k-algebra morphism

Prop For a k-module V, the tensor algebra $T(V) = H$ is a Hopf algebra with

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$

$$\varepsilon(v) = 0 \qquad v \in V$$

$$S(v) = -v$$

pf Recall $T(V)$ is a bialgebra with above Δ, ε

Consider S:

$\exists$ k-module hom

$$V \longrightarrow T(V)^{op}$$
$$v \mapsto -v$$

This extends to a k-alg morphism

$$S : T(V) \to T(V)^{op}$$

by univ property of $T(V)$

By constr

$$S(ab) = S(b)S(a) \qquad a, b \in T(V)$$

$$S(1_H) = 1_H \qquad H = T(V)$$

$$S(v) = -v \qquad v \in V$$

Show S is an antipode for $T(V)$:

By constr

$$S : T(V) \to T(V) \quad \text{is } k\text{-module hom } \checkmark$$

Show: $\forall a \in T(V)$

$$\varepsilon(a)\, 1_H = \sum_{(a)} a_1 S(a_2) \qquad (*)$$

<u>claim 1</u> $(*)$ holds for $a = 1_H$

$$\Delta(1_H) = 1_H \otimes 1_H$$

$(*)$ becomes

$$\underbrace{\varepsilon(1_H)}_{=1}\, 1_H \overset{?}{=} 1_H \underbrace{S(1_H)}_{=1_H}$$

ok

<u>claim 2</u> $(*)$ holds for $a = v \in V$

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$

$(*)$ becomes

$$\underbrace{\varepsilon(v)}_{=0}\, 1_H \overset{?}{=} \underbrace{v \underbrace{S(1_H)}_{=1_H} + 1_H \underbrace{S(v)}_{=-v}}_{=0}$$

ok

<u>claim 3</u> Assume $(*)$ holds for $a, b \in T(V)$.

Then $(*)$ hold for ab.

Recall
$$\Delta(ab) = \Delta(a)\,\Delta(b)$$
$$= \sum_{(a)} \sum_{(b)} a_1 b_1 \otimes a_2 b_2$$

$(*)$ becomes

$$\underset{\displaystyle \overset{\shortparallel}{\varepsilon(a)\,\varepsilon(b)}}{\varepsilon(ab)\,1_H} \overset{?}{=} \underbrace{\sum_{(a)} \sum_{(b)} a_1 b_1 \underset{\displaystyle \overset{\shortparallel}{S(b_2)\,S(a_2)}}{S(a_2 b_2)}}$$

$$\shortparallel$$

$$\underbrace{\sum_{(a)} a_1 \Big(\underbrace{\sum_{(b)} b_1 S(b_2)}_{\overset{\shortparallel}{\varepsilon(b)\,1_H}} \Big) S(a_2)}$$

$$\shortparallel$$

$$\underbrace{\varepsilon(b) \underbrace{\sum_{(a)} a_1 S(a_2)}_{\overset{\shortparallel}{\varepsilon(a)\,1_H}}}$$

$$\shortparallel$$

$$\varepsilon(a)\,\varepsilon(b)\,1_H$$

OK

(*) holds by claims 1-3 and since V generates the algebra $T(V)$.

Similarly we have

$$\varepsilon(a)\, 1_H = \sum_{(a)} S(a_1)\, a_2 \qquad a \in T(V)$$

So S is antipode that turns the bialgebra $T(V)$ into a Hopf algebra. $\square$

LEM For a k-module V, the symmetric algebra $\underset{H}{\underset{\shortparallel}{\mathrm{Sym}(V)}}$ is a Hopf algebra with

$$\Delta(v) = v\otimes 1 + 1\otimes v, \qquad v\in V$$
$$\varepsilon(v) = 0$$
$$S(v) = -v$$

pf Recall

$$\mathrm{Sym}(V) = T(V)/J$$

2-sided ideal J of $T(V)$ is generated by

$$uv - vu, \qquad u,v\in V$$

Consider antipode of $T(V)$

$$S: T(V)\to T(V)$$

Apply S to J:

For $u,v\in V$

$$S(uv-vu) = \underset{-v}{\underset{\shortparallel}{S(v)}}\,\underset{-u}{\underset{\shortparallel}{S(u)}} - \underset{-u}{\underset{\shortparallel}{S(u)}}\,\underset{-v}{\underset{\shortparallel}{S(v)}}$$

$$= vu - uv$$

$$\in J$$

More generally for $a, b \in T(V)$

$$S(a(uv-vu)b) = S(b)\underbrace{S(uv-vu)}_{\in J}S(a)$$

$$\in J$$

So $S(J) \subseteq J$

Consider the k-module hom

$$T(V) \xrightarrow{S} T(V) \xrightarrow{\text{can}} Sym(V) \qquad (*)$$

$$x \longrightarrow x + J$$

$(*)$ sends $J \to 0$

So $(*)$ induces a k-module hom

$$S : Sym(V) \to Sym(V)$$

By constr $S(ab) = S(b)S(a) = S(a)S(b)$ $\quad a, b \in T(V)$

$$S(v) = -v \qquad v \in V$$

$$S(1_H) = 1_H \qquad H = Sym(V)$$

So $S : Sym(V) \to Sym(V)$

is K-alg morphism.

Show S is antipode.

S makes these diagrams commute:

$$\begin{tikzcd} H\otimes H \arrow[r, "1\otimes S"] & H\otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & K \arrow[r, "u"'] & H \end{tikzcd}$$

$$\begin{tikzcd} H\otimes H \arrow[r, "S\otimes 1"] & H\otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & K \arrow[r, "u"'] & H \end{tikzcd}$$

because all maps involved are K-algebra morphisms and for $v \in V$,

$$\begin{tikzcd} v\otimes 1 + 1\otimes v \arrow[r] & v\otimes 1 + 1\otimes(-v) \arrow[d] \\ v \arrow[u] \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} v\otimes 1 + 1\otimes v \arrow[r] & (-v)\otimes 1 + 1\otimes v \arrow[d] \\ v \arrow[u] \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

$\square$

Prop For a group G

the group algebra $\underset{H}{\overset{}{kG}}$ is a Hopf algebra with

$$\Delta(t_g) = t_g \otimes t_g$$

$$\varepsilon(t_g) = 1 \qquad g \in G$$

$$S(t_g) = t_{g^{-1}}$$

pf Recall kG is a bialgebra with above Δ, ε.

Consider S.

$\exists$ k-module hom

$$S: \begin{aligned} kG &\to kG \\ t_g &\to t_{g^{-1}} \end{aligned}$$

Show: $\forall a \in kG$

$$\varepsilon(a)\, 1_H \overset{!}{=} \sum_{(a)} a_1 S(a_2) \qquad (*)$$

WLOG $a = t_g \qquad g \in G$

$$\Delta(t_g) = t_g \otimes t_g$$

(*) becomes

$$\underbrace{\varepsilon(t_g)}_{\underset{1}{\parallel}} 1_H \overset{?}{=} \underbrace{t_g \underbrace{S(t_g)}_{\underset{t_{g^{-1}}}{\parallel}}}_{\underset{\underset{\underset{1_H}{\parallel}}{\underset{t_e}{\parallel}}}{\underset{t_{gg^{-1}}}{\parallel}}}$$

ok

(*) is verified.

Similarly one checks

$$\varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2 \qquad \forall a \in kG$$

So S is an antipode that turns the bialgebra kG into a Hopf algebra $\square$

Prop Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Then H is a Hopf algebra, and its antipode

S satisfies

$$S(H_n) \subseteq H_n \quad \text{for } n \in \mathbb{N}$$

pf <u>Claim 1</u> $\exists$ k module hom

$$S : H \to H$$

st both

$$\varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2 \qquad a \in H \qquad *$$

$$S(H_n) \subseteq H_n \qquad n \in \mathbb{N}$$

<u>pf cl 1</u> We define

$$S|_{H_n} : H_n \to H_n$$

by ind on n

<u>$n = 0$:</u> $S(1_H) = 1_H$

<u>$n \geq 1$:</u> Recall for $a \in H_n$ $\varepsilon(a) = 0$ so $*$

becomes

$$0 = \sum_{(a)} S(a_1) a_2 \qquad **$$

Recall

$$\Delta(a) - a\otimes 1 - 1\otimes a \in \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i}$$

So ** becomes

$$0 - S(a)1_H - S(1_H)a \in \underbrace{\bigoplus_{i=1}^{n-1} \overbrace{S(H_i)}^{\substack{H_i \\ \cup\!\!\!/}} H_{n-i}}_{\substack{\cap \\ H_n}}$$

So

$S(a) + a$ is an element of H_n that is uniquely det by the action of S on $H_1, H_2, \ldots, H_{n-1}$

<u>Claim 2</u> $\exists$ k-module hom

$$\tilde{S} : H \to H$$

st both

$$\varepsilon(a)1_H = \sum_{(a)} a_1 \tilde{S}(a_2) \qquad a \in H$$

$$\tilde{S}(H_n) \subseteq H_n \qquad n \in \mathbb{N}$$

<u>pf cl 2</u> sim to cl 1

<u>claim 3</u> $S = \tilde{S}$

<u>pf cl 3</u> Recall Convolution algebra End (H) with mult $\star$ and ident $\mathbb{1} : H \to H$, $a \to \varepsilon(a) 1_H$

By cl 1

$$\underline{\mathbb{1}} = S \star id$$

By cl 2

$$\underline{\mathbb{1}} = id \star \tilde{S}$$

Now

$$\begin{aligned} S &= S \star \mathbb{1} \\ &= S \star id \star \tilde{S} \\ &= \mathbb{1} \star \tilde{S} \\ &= \tilde{S} \end{aligned}$$

We have shown S is an antipode. Result follows. $\square$