

Lec 6 Monday Sept 19

9/19/16

1

Given a bialgebra  $H$

Recall an antipode for  $H$  is a  $K$ -module hom

$$S: H \rightarrow H$$

such that for  $a \in H$ ,

$$\sum_{(a)} a_1 S(a_2) = \varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2$$

In terms of commuting diagrams, this is

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H \\
 \Delta \uparrow & & \downarrow m \\
 H & \xrightarrow[\varepsilon]{\kappa} & H
 \end{array}$$

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H \\
 \Delta \uparrow & & \downarrow m \\
 H & \xrightarrow[\varepsilon]{\kappa} & H
 \end{array}$$

Going to show

- $S$  is unique if it exists

- $S(1_H) = 1_H$

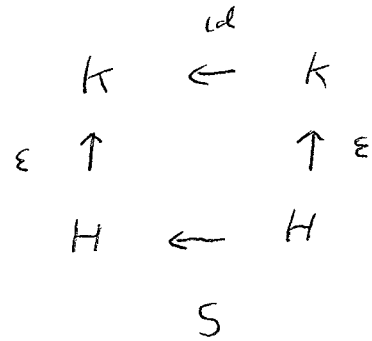
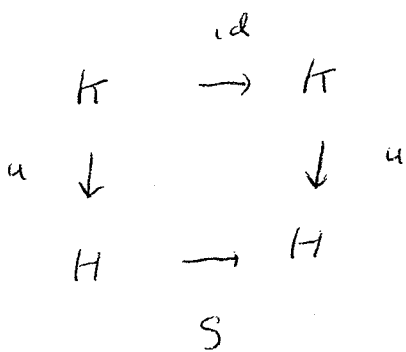
- $\varepsilon(S(a)) = \varepsilon(a)$

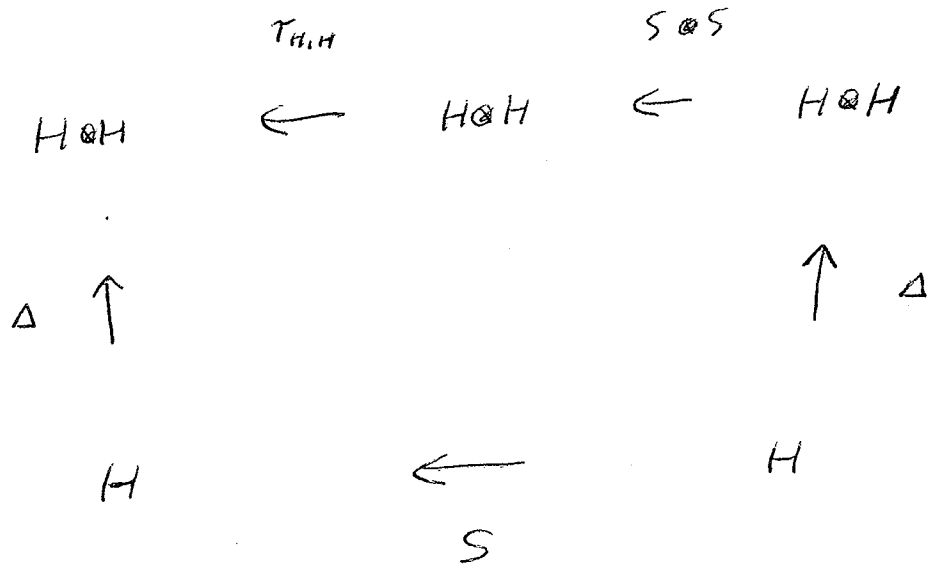
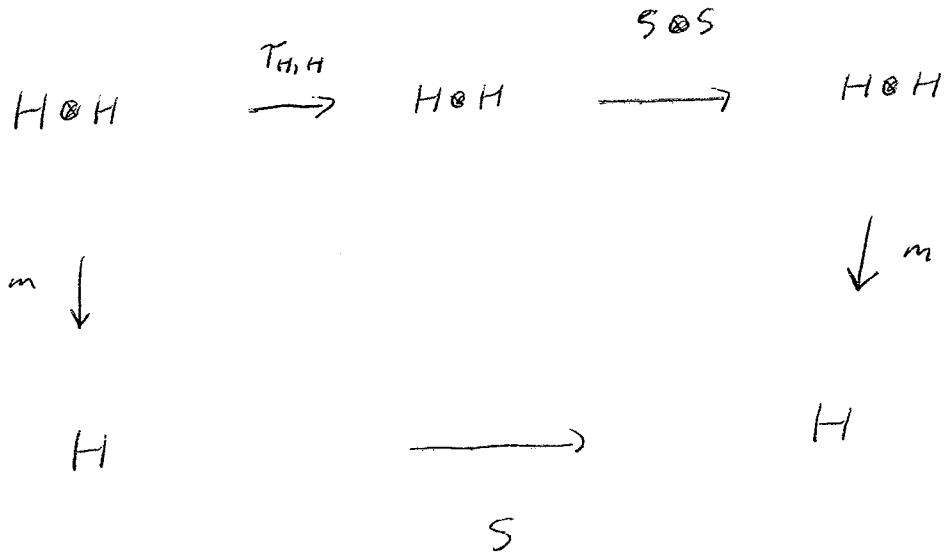
$a, b \in H$

- $S(ab) = S(b)S(a)$

- $\Delta(S(a)) = \sum_{(a)} S(a_2) \otimes S(a_1)$

In terms of diagrams this is





LEM For an antipode  $S$  of a bialg  $H$

$$S(1_H) = 1_H$$

pf  $\Delta, \varepsilon$  are alg morphisms so

$$\Delta(1_H) = 1_H \otimes 1_H$$

$$\varepsilon(1_H) = 1$$

For  $x = 1_H$

$$\varepsilon(x) 1_H = \sum_{(x)} S(x_1) x_2$$

becomes

$$1_H = S(1_H) 1_H$$



In order to obtain the remaining properties of  $S_1$ , it is helpful to bring in the convolution product

Notation For  $K$ -modules  $U, V$   
 $\text{Hom}(U, V)$  is the set of  $K$ -module homs  $U \rightarrow V$

Write  $\text{End}(V) = \text{Hom}(V, V)$

Given a  $K$ -algebra  $A$  and  $K$ -coalgebra  $C$   
 we define a product  $\star$  on  $\text{Hom}(C, A)$  as follows.

$\forall f, g \in \text{Hom}(C, A)$

$$(f \star g)(c) = \sum_{(c)} f(c_1) g(c_2) \quad \text{"convolution product"}$$

One checks

$$f \star g \in \text{Hom}(C, A)$$

LEM With above notation,

$\star$  makes  $\text{Hom}(C, A)$  a  $k$ -algebra with identity

"convolution algebra"

$$\begin{aligned} \mathbb{1} &: C \xrightarrow{\epsilon} k \xrightarrow{\eta} A \\ c &\rightarrow \epsilon(c) \rightarrow \epsilon(c)\eta \end{aligned}$$

pf  $\text{Hom}(C, A)$  is  $k$ -module ✓

$\star$  is  $k$ -linear in each argument ✓

$\star$  is assoc: For  $f, g, h \in \text{Hom}(C, A)$

$$(f \star g) \star h \stackrel{?}{=} f \star (g \star h)$$

$\forall c \in C$  each side at  $c$  is

$$\sum_{(c)} f(c_1) g(c_2) h(c_3)$$

$\star$  is assoc ✓

check  $\mathbb{1}$  is identity

$\forall f \in \text{Hom}(C, A)$

$$\mathbb{1} \star f \stackrel{?}{=} f \stackrel{?}{=} f \star \mathbb{1}$$

$\forall c \in C$

$$\sum_{(c)} \epsilon(c_1) f(c_2) \stackrel{?}{=} f(c) \stackrel{?}{=} \sum_{(c)} f(c_1) \epsilon(c_2)$$

yes since  $\sum_{(c)} \epsilon(c_1) c_2 = c = \sum_{(c)} c_1 \epsilon(c_2)$

□

Given bialgebra  $H$

Consider convolution algebra

$$\text{Hom}(C, A) \quad C = H, \quad A = H$$

So

$$\text{Hom}(C, A) = \text{End}(H)$$

Consider the identity map

$$\text{id} \in \text{End}(H)$$

Prop With above notation, for  $S \in \text{End}(H)$  TFAE =

(i)  $S$  is an antipode for  $H$

(ii)  $S$  is the inverse of  $\text{id}$  in the convolution algebra

pf: (i)  $\Leftrightarrow$

$$\sum_{(a)} S(a_2) a_2 = \sum_{(a)} \mathbb{1}(a) 1_H = \sum_{(a)} a_1 S(a_2) \quad \forall a \in H$$

$$\parallel \qquad \parallel \qquad \parallel$$

$$(S \star id)(a) \qquad \mathbb{1}(a) \qquad (id \star S)(a)$$

$\Leftrightarrow$

$$S \star id = \mathbb{1} = id \star S$$

$\Leftrightarrow$  (ii)

□



COR For a bialgebra  $H$ ,

the antipode  $S$  is unique if it exists.

pf In a  $K$ -algebra any element has

at most one inverse.  $\square$

DEF A Hopf algebra is a bialgebra

that has an antipode.

LEM For a Hopf algebra  $H$  with antipode  $S$

$$S(ab) = S(b)S(a) \quad a, b \in H$$

pf we have

$$\begin{aligned} \Delta(ab) &= \Delta(a) \Delta(b) \\ &\parallel \parallel \\ &= \sum_{(a)} a_1 \otimes a_2 \quad \sum_{(b)} b_1 \otimes b_2 \\ &= \sum_{(a)} \sum_{(b)} a_1 b_1 \otimes a_2 b_2 \end{aligned}$$

So

$$\begin{aligned} \sum_{(a)} \sum_{(b)} S(a_1 b_1) a_2 b_2 &= \varepsilon(ab) 1_H \\ &= \varepsilon(a) \varepsilon(b) 1_H \\ &= \varepsilon(a) \sum_{(b)} S(b_1) b_2 \\ &= \sum_{(b)} S(b_1) \underbrace{\varepsilon(a)}_{\sum_{(a)} S(a_1) a_2} b_2 \\ &= \sum_{(a)} \sum_{(b)} S(b_1) S(a_1) a_2 b_2 \end{aligned}$$

So

$$0 = \sum_{(a)} \sum_{(b)} \left( \zeta(a, b) - \zeta(b) \zeta(a) \right) a_2 b_2$$

$$= \sum_{(b)} \left( \sum_{(a)} \left( \zeta(a, b) - \zeta(b) \zeta(a) \right) a_2 \right) b_2$$

$$\left[ \begin{array}{ccc} & H & \longrightarrow & H \\ \text{def } \varphi_a : & & & \\ x \rightarrow & \sum_{(a)} & \left( \zeta(a, x) - \zeta(x) \zeta(a) \right) a_2 & \end{array} \right]$$

$$= \sum_{(b)} \varphi_a(b) b_2$$

$$= (\varphi_a * id)(b)$$

So  $\varphi_a * id = 0$

So  $\varphi_a = 0$

So

$$0 = \varphi_a(b)$$

$$= \sum_{(a)} (s(a,b) - s(b)s(a)) a_2$$

$$\left[ \text{def } \phi_b: \begin{array}{ccc} H & \longrightarrow & H \\ x & \longrightarrow & s(xb) - s(b)s(x) \end{array} \right]$$

$$= \sum_{(a)} \phi_b(a_1) a_2$$

$$= (\phi_b \star \text{id})(a)$$

So  $\phi_b \star \text{id} = 0$

So  $\phi_b = 0$

Now

$$0 = \phi_b(a)$$

$$= s(a)b - s(b)s(a)$$



LEM For a Hopf algebra  $H$  with antipode  $S$ ,

$$\Delta(S(a)) = \sum_{(a)} S(a_2) \otimes S(a_1) \quad a \in H$$

pf Consider convolution algebra

$$\text{Hom}(C, A) \quad C = H, \quad A = H \otimes H$$

Identify is

$$\begin{array}{ccccc} H & \longrightarrow & K & \longrightarrow & H \otimes H \\ \parallel & & & & \\ a & \longrightarrow & \varepsilon(a) & \longrightarrow & \varepsilon(a) 1_H \otimes 1_H \end{array}$$

Consider these elements in  $\text{Hom}(C, A)$ :

$$\Delta : H \rightarrow H \otimes H \tag{1}$$

$$\begin{array}{ccccc} H & \longrightarrow & H & \longrightarrow & H \otimes H \\ & & S & & \Delta \end{array} \tag{2}$$

$$\begin{array}{ccccccc} H & \longrightarrow & H \otimes H & \longrightarrow & H \otimes H & \longrightarrow & H \otimes H \\ & & \Delta & & S \otimes S & & \tau_{H,H} \end{array} \tag{3}$$

wish to show (2) = (3)

To do this, show each of (2), (3) is the inverse of (1) in the convolution algebra (ex)

□

LEM For a Hopf algebra  $H$  with antipode  $S$

$$\varepsilon(S(a)) = \varepsilon(a) \quad a \in H$$

pf We have

$$\varepsilon(a) 1_H = \sum_{(a)} a_1 S(a_2) = \sum_{(a)} S(a_1) a_2$$

Also for  $b = S(a)$

$$\varepsilon(b) 1_H = \sum_{(b)} b_1 S(b_2)$$

"

$$\varepsilon(S(a)) \quad \left[ \Delta(b) = \sum_{(a)} S(a_2) \otimes S(a_1) \right]$$

$$= \sum_{(a)} S(a_2) S^2(a_1)$$

$$= S \left( \sum_{(a)} S(a_1) a_2 \right)$$

$$= S \left( \varepsilon(a) 1_H \right)$$

$$= \varepsilon(a) S(1_H)$$

$$= \varepsilon(a) 1_H$$

$S_0$

$$\varepsilon(S(a)) = \varepsilon(a)$$

□