Lec 6 Monday Sept 19

Given a bialgebra H

Recall an <u>antipode</u> for H is a k-module hom

$$S : H \to H$$

such that for $a \in H$,

$$\sum_{(a)} a_1 S(a_2) = \varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2$$

In terms of commuting diagrams, this is

$$\begin{tikzcd} H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & k \arrow[r, "u"'] & H \end{tikzcd}$$

$$\begin{tikzcd} H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & k \arrow[r, "u"'] & H \end{tikzcd}$$

Going to show

- S is unique if it exists
- $S(1_H) = 1_H$
- $\varepsilon(S(a)) = \varepsilon(a)$
- $S(ab) = S(b)S(a)$ $\quad a, b \in H$
- $\Delta(S(a)) = \sum_{(a)} S(a_2) \otimes S(a_1)$

In terms of diagrams this is

$$\begin{tikzcd} K \arrow[r, "id"] \arrow[d, "u"'] & K \arrow[d, "u"] \\ H \arrow[r, "S"'] & H \end{tikzcd} \qquad \begin{tikzcd} K & K \arrow[l, "id"'] \\ H \arrow[u, "\varepsilon"] & H \arrow[l, "S"] \arrow[u, "\varepsilon"'] \end{tikzcd}$$

$$\begin{tikzcd}
H \otimes H \arrow[r, "\tau_{H,H}"] \arrow[d, "m"'] & H \otimes H \arrow[r, "S \otimes S"] & H \otimes H \arrow[d, "m"] \\
H \arrow[rr, "S"'] & & H
\end{tikzcd}$$

$$\begin{tikzcd}
H \otimes H & H \otimes H \arrow[l, "\tau_{H,H}"'] & H \otimes H \arrow[l, "S \otimes S"'] \\
H \arrow[u, "\Delta"] & & H \arrow[ll, "S"] \arrow[u, "\Delta"']
\end{tikzcd}$$

LEM For an antipode S of a bialg H

$$S(1_H) = 1_H$$

pf Δ, ε are alg morphisms so

$$\Delta(1_H) = 1_H \otimes 1_H$$

$$\varepsilon(1_H) = 1$$

For $x = 1_H$

$$\varepsilon(x)\, 1_H = \sum_{(x)} S(x_1)\, x_2$$

becomes

$$1_H = S(1_H)\, 1_H$$

$\square$

In order to obtain the remaining properties of S, it is helpful to bring in the convolution product

<u>Notation</u> For k-modules U, V

$\mathrm{Hom}(U,V)$ is the set of k-module homs $U \to V$

Write $\mathrm{End}(V) = \mathrm{Hom}(V,V)$

Given a k-algebra A and k-coalg C

we define a product $\star$ on $\mathrm{Hom}(C,A)$ as follows.

$\forall\ f, g \in \mathrm{Hom}(C,A)$

$$(f \star g)(c) = \sum_{(c)} f(c_1)\, g(c_2)$$ "convolution product"

One checks

$$f \star g \in \mathrm{Hom}(C,A)$$

LEM With above notation, $\star$ makes $Hom(C,A)$ a k-algebra with identity "Convolution algebra"

$$\mathbb{1} : \quad C \xrightarrow{\varepsilon} k \xrightarrow{u} A$$
$$c \to \varepsilon(c) \to \varepsilon(c) 1_A$$

pf $Hom(C,A)$ is k-module ✓

$\star$ is k-linear in each argument ✓

$\star$ is assoc: For $f, g, h \in Hom(C,A)$

$$(f \star g) \star h \stackrel{?}{=} f \star (g \star h)$$

$\forall c \in C$ each side at c is

$$\sum_{(c)} f(c_1)\, g(c_2)\, h(c_3)$$ ✓

$\star$ is assoc ✓

check $\mathbb{1}$ is ident: $\forall f \in Hom(C,A)$

$$\mathbb{1} \star f \stackrel{?}{=} f \stackrel{?}{=} f \star \mathbb{1}$$

$\forall c \in C$

$$\sum_{(c)} \varepsilon(c_1) f(c_2) \stackrel{?}{=} f(c) \stackrel{?}{=} \sum_{(c)} f(c_1)\, \varepsilon(c_2)$$

yes since $\sum_{(c)} \varepsilon(c_1) c_2 = c = \sum_{(c)} c_1 \varepsilon(c_2)$

$\square$

Given bialgebra H

Consider convolution algebra

$Hom(C, A)$ $\qquad C = H, \qquad A = H$

So

$$Hom(C, A) = End(H)$$

Consider the identity map

$$id \in End(H)$$

Prop With above notation, for $S \in End(H)$ TFAE:

(i) S is an antipode for H

(ii) S is the inverse of id in the convolution algebra pf: (i) $\Longleftrightarrow$

$$\sum_{(a)} S(a_1)a_2 = \varepsilon(a)1_H = \sum_{(a)} a_1 S(a_2) \quad \forall a \in H$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ (S \star id)(a) & & \mathbb{1}(a) & & (id \star S)(a) \end{array}$$

$\Longleftrightarrow$

$$S \star id = \mathbb{1} = id \star S$$

$\Longleftrightarrow$ (ii) $\square$

COR For a bialgebra H,

the antipode S is unique if it exists.

pf In a K-algebra any element has

at most one inverse. $\square$

DEF A <u>Hopf algebra</u> is a bialgebra

that has an antipode.

LEM For a Hopf algebra H with antipode S

$$S(ab) = S(b)S(a) \qquad a, b \in H$$

pf We have

$$\Delta(ab) = \underbrace{\Delta(a)}_{\sum_{(a)} a_1 \otimes a_2} \; \underbrace{\Delta(b)}_{\sum_{(b)} b_1 \otimes b_2}$$

$$= \sum_{(a)} \sum_{(b)} a_1 b_1 \otimes a_2 b_2$$

So

$$\sum_{(a)} \sum_{(b)} S(a_1 b_1)\, a_2 b_2 = \varepsilon(ab)\, 1_H$$

$$= \varepsilon(a)\,\varepsilon(b)\, 1_H$$

$$= \varepsilon(a) \sum_{(b)} S(b_1)\, b_2$$

$$= \sum_{(b)} S(b_1) \underbrace{\varepsilon(a)}_{= \sum_{(a)} S(a_1) a_2} b_2$$

$$= \sum_{(a)} \sum_{(b)} S(b_1) S(a_1)\, a_2 b_2$$

So

$$0 = \sum_{(a)} \sum_{(b)} \Big(S(a_1 b_1) - S(b_1) S(a_1) \Big) \; a_2 b_2$$

$$= \sum_{(b)} \left(\sum_{(a)} \Big(S(a_1 b_1) - S(b_1) S(a_1) \Big) \; a_2 \right) b_2$$

$$\left[\text{def} \quad \varphi_a : \begin{array}{ccc} H & \longrightarrow & H \\ x & \mapsto & \sum_{(a)} \big(S(a_1 x) - S(x) S(a_1) \big) \, a_2 \end{array} \right]$$

$$= \sum_{(b)} \varphi_a(b_1) \, b_2$$

$$= (\varphi_a \star id)(b)$$

So $\quad \varphi_a \star id = 0$

So $\quad \varphi_a = 0$

So

$$0 = \varphi_a(b)$$

$$= \sum_{(a)} \left(S(a_1 b) - S(b) S(a_1) \right) a_2$$

$$\left[\text{def } \phi_b : \begin{array}{ccc} H & \longrightarrow & H \\ x & \rightarrow & S(xb) - S(b) S(x) \end{array} \right]$$

$$= \sum_{(a)} \phi_b(a_1) a_2$$

$$= (\phi_b \star id)(a)$$

So $\phi_b \star id = 0$

So $\phi_b = 0$

Now

$$0 = \phi_b(a)$$

$$= S(ab) - S(b) S(a)$$

$\square$

LEM For a Hopf algebra H with antipode S,

$$\Delta(S(a)) = \sum_{(a)} S(a_2) \otimes S(a_1) \qquad a \in H$$

pf Consider convolution algebra

$$\mathrm{Hom}(C, A) \qquad C = H, \qquad A = H \otimes H,$$

Identity is

$$\mathbb{1} : \begin{array}{ccccc} H & \longrightarrow & k & \longrightarrow & H \otimes H \\ a & \longrightarrow & \varepsilon(a) & \longrightarrow & \varepsilon(a)\, 1_H \otimes 1_H \end{array}$$

Consider these elements in $\mathrm{Hom}(C, A)$:

$$\Delta : H \to H \otimes H \tag{1}$$

$$H \xrightarrow{S} H \xrightarrow{\Delta} H \otimes H \tag{2}$$

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{S \otimes S} H \otimes H \xrightarrow{\tau_{H,H}} H \otimes H \tag{3}$$

Wish to show $(2) = (3)$

To do this show each of (2), (3) is the inverse of (1) in the convolution algebra (ex)

$\square$

LEM For a Hopf algebra H with antipode S

$$\varepsilon(S(a)) = \varepsilon(a) \qquad a \in H$$

pf We have

$$\varepsilon(a)\, 1_H = \sum_{(a)} a_1 S(a_2) = \sum_{(a)} S(a_1)\, a_2$$

Also for $b = S(a)$

$$\varepsilon(b)\, 1_H = \sum_{(b)} b_1 S(b_2)$$

$$\| $$

$$\varepsilon(S(a)) \qquad \left[\ \Delta(b) = \sum_{(a)} S(a_2) \otimes S(a_1)\ \right]$$

$$= \sum_{(a)} S(a_2)\, S^2(a_1)$$

$$= S\left(\sum_{(a)} S(a_1)\, a_2 \right)$$

$$= S\left(\varepsilon(a)\, 1_H \right)$$

$$= \varepsilon(a)\, S(1_H)$$

$$= \varepsilon(a)\, 1_H$$

So

$$\varepsilon(S(a)) = \varepsilon(a)$$

$\square$