Lec 5 Friday Sept 16

Given a k-coalgebra C

Recall a coideal of C is a

k-submodule J of C st

$$\Delta(J) \subseteq C \otimes J + J \otimes C,$$

$$\varepsilon(J) = 0$$

Ex Recall tensor alg $T(V) = C$ is coalg.

Take $J =$ 2-sided ideal gen by

$uv - vu \quad u, v \in V$

then J is a coideal of $T(V)$

pf For $x \in J$ show

$$\Delta(x) \in C \otimes J + J \otimes C$$

WLOG $\quad x = y(uv - vu)z \qquad y, z \in C, \; u, v \in V$

$$\Delta(x) = \Delta(y) \underbrace{\Delta(uv - vu)}_{= (uv-vu)\otimes 1 + 1 \otimes (uv - vu)} \Delta(z)$$

$$\subseteq C \otimes J + J \otimes C$$

Also show

$$\varepsilon(x) = 0$$

$$\varepsilon(x) = \varepsilon(y) \underbrace{\varepsilon(uv - vu)}_{= 0} \varepsilon(z)$$

$$= 0$$

$\square$

Graded algebras and coalgebras

Given k-module V

A *grading* of V is a sequence $\{V_n\}_{n\in\mathbb{N}}$ of

k-submodules whose direct sum is V

Possibly $V_n = 0$ for some n

Ex. Take $V = k$

$$K_0 = k$$
$$K_n = 0 \qquad n = 1, 2, \ldots$$

Given k-modules U, V Consider k-module $U \otimes V$.

Given gradings $\{U_n\}_{n\in\mathbb{N}}$ of U
$\{V_n\}_{n\in\mathbb{N}}$ of V

For $n \in \mathbb{N}$ define

$$(U\otimes V)_n = U_0 \otimes V_n + U_1 \otimes V_{n-1} + \cdots + U_n \otimes V_0$$
$$\subseteq U \otimes V$$

Then

$$\{(U\otimes V)_n\}_{n\in\mathbb{N}}$$

is a grading of $U\otimes V$

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Given k-algebra A

Given grading $\{A_n\}_{n\in\mathbb{N}}$ of A

Call this an <u>algebra grading</u> whenever

$$A_i\, A_j \subseteq A_{i+j} \qquad i,j\in\mathbb{N}$$

$$1_A \in A_0$$

LEM For above A TFAE

(i) $\{A_n\}_{n\in\mathbb{N}}$ is an algebra grading

(ii) $\forall n \in \mathbb{N}$

$$m\left((A\otimes A)_n\right) \subseteq A_n$$

$$u\left(k_n\right) \subseteq A_n$$

pf (i) → (ii) For $0 \le i \le n$

$$m(A_i \otimes A_{n-i}) = A_i A_{n-i} \subseteq A_n$$

Also

$$u(k_0) = u(k) \subseteq k 1_A \subseteq A_0$$

and for $n \ge 1$

$$u(k_n) = u(0) = 0 \subseteq A_n$$

(ii) → (i)

For $i, j \in \mathbb{N}$

$$A_i A_j = m(A_i \otimes A_j) \subseteq m((A \otimes A)_{i+j}) \subseteq A_{i+j}$$

$$1_A = u(1) \in u(k_0) \subseteq A_0$$

$\square$

The above Lemma suggests how to define a coalgebra grading.

Given coalgebra C

Given grading $\{C_n\}_{n\in\mathbb{N}}$ of C

Call this a <u>coalgebra grading</u> whenever

$$\Delta(C_n) \subseteq (C\otimes C)_n \qquad n\in\mathbb{N}$$

$$\varepsilon(C_n) \subseteq K_n$$

LEM With above notation TFAE

(i) $\{C_n\}_{n\in\mathbb{N}}$ is a coalgebra grading

(ii) $\forall\, n\in\mathbb{N}$

$$\Delta(C_n) \subseteq C_0\otimes C_n + C_1\otimes C_{n-1} + \cdots + C_n\otimes C_0$$

$$\varepsilon(C_n) = 0 \quad \text{if } n\geq 1.$$

pf Routine $\square$

Given a bialgebra H

Given a grading $\{H_n\}_{n \in \mathbb{N}}$ of H

Call this a <u>bialgebra grading</u> whenever

it is both an algebra and coalgebra grading.

Ex Given K-module V

(i) For the tensor algebra $T(V)$ the grading

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

is a bialgebra grading

(ii) For the symmetric algebra $S(V)$ the grading

$$S(V) = \sum_{n \in \mathbb{N}} \mathrm{Sym}^n(V)$$

is a bialgebra grading

LEM Given graded bialgebra $H = \bigoplus_{n \in \mathbb{N}} H_n$

Then for the set P of primitive elements

(i) $P = \bigoplus_{n \in \mathbb{N}} (P \cap H_n)$

(ii) P is a coideal of H

pf (i) Fix $x \in P$

write $x = \sum_{n \in \mathbb{N}} x_n$ $\quad x_n \in H_n$

Fix $n \in \mathbb{N}$ show $x_n \in P$

We have $\Delta(x) = x \otimes 1 + 1 \otimes x$

So
$$0 = x \otimes 1 + 1 \otimes x - \Delta(x)$$
$$= \sum_{n \in \mathbb{N}} \underbrace{\left(x_n \otimes 1 + 1 \otimes x_n - \Delta(x_n)\right)}_{\in (H \otimes H)_n}$$

So each summand is 0

$\Delta(x_n) = x_n \otimes 1 + 1 \otimes x_n$

so $x_n \in P$

(ii) show

$$\Delta(P) \subseteq P \otimes H + H \otimes P,$$

$$\varepsilon(P) = 0.$$

For $x \in P$

$$\Delta(x) = \underbrace{x \otimes 1}_{\in P \otimes H} + \underbrace{1 \otimes x}_{\in H \otimes P}$$

Also using

$$\begin{tikzcd} k \otimes H & H \otimes H \arrow[l, "\varepsilon \otimes id"'] \\ & H \arrow[u, "\Delta"'] \arrow[ul, leftrightarrow, "can"] \end{tikzcd}$$

$$\begin{tikzcd} \varepsilon(x) \otimes 1 + 1 \otimes x = 1 \otimes x & x \otimes 1 + 1 \otimes x \arrow[l] \\ & x \arrow[u] \arrow[ul, leftrightarrow] \end{tikzcd} \qquad \varepsilon(1) = 1$$

So

$$\varepsilon(x) = 0$$

$\square$

Given graded algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$

Obs A_0 is a K-subalgebra of A,

$u(K) \subseteq A_0$

Call the grading <u>connected</u> whenever

$u(K) = A_0$

LEM Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

(i) the maps

$$u : k \to H_0 \qquad \varepsilon|_{H_0} : H_0 \to k$$

are inverses and hence bijections

(ii)
$$\ker(\varepsilon) = \underbrace{\bigoplus_{n \geq 1} H_n}_{\overset{\text{def}}{=} I}$$

(iii) For $n \geq 1$ and $x \in H_n$

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i}$$

(iv) For $x \in H$

$$\Delta(x) - x \otimes 1 \in H \otimes I$$

$$\Delta(x) - 1 \otimes x \in I \otimes H$$

(v) For $x \in I$

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in I \otimes I$$

pf (i) Since u, ε are k-linear and

$\varepsilon(1_H) = 1$, $u(1) = 1_H$

(ii) $\supseteq$: $\varepsilon(H_n) = 0$ for $n \geq 1$ since coalg H is graded

$\subseteq$ For $x \in \ker(\varepsilon)$ show $x \in I$

write $x = \sum_{n \in \mathbb{N}} x_n$ $x_n \in H_n$

$$0 = \varepsilon(x) = \sum_{n \in \mathbb{N}} \varepsilon(x_n)$$

$\varepsilon(x_n) = 0$ for $n \geq 1$

$$= \varepsilon(x_0)$$

Now $x_0 = 0$ by (i)

So $x = \sum_{i=1}^{n} x_i \in I$

(iii) Since coalg H is graded

$$\Delta(H_n) \subseteq \bigoplus_{i=0}^{n} H_i \otimes H_{n-i}$$

$$= \underbrace{\overset{k1_H}{\overset{\shortparallel}{H_0}} \otimes H_n}_{\overset{\shortparallel}{1_H \otimes H_n}} + \underbrace{H_n \otimes \overset{k1_H}{\overset{\shortparallel}{H_0}}}_{\overset{\shortparallel}{H_n \otimes 1_H}} + \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i}$$

So

$$\Delta(x) - 1 \otimes \underset{\underset{H_n}{\uparrow}}{\varphi(x)} - \underset{\underset{H_n}{\uparrow}}{\phi(x)} \otimes 1 \in \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i} \qquad (*)$$

show $\varphi(x) = x = \phi(x)$

Using

$$x = \sum_{(x)} \varepsilon(x_1)\, x_2$$

$(*)$ gives

$$x - \underset{\underset{1}{\shortparallel}}{\varepsilon(1)}\, \varphi(x) - \underbrace{\varepsilon(\phi(x))}_{\overset{\shortparallel}{0}} 1 \in \bigoplus_{i=1}^{n-1} \underbrace{\varepsilon(H_i)}_{\overset{\shortparallel}{0}} H_{n-i}$$

so $x = \varphi(x)$

Using

$$x = \sum_{(x)} x_1\, \varepsilon(x_2)$$

$(*)$ yields

$$x = \phi(x)$$

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(iv) WLOG $x \in H_n$

For $n=0$

$$\Delta(x) - x \otimes 1 = 0 \in H \otimes I$$

$$\Delta(x) - 1 \otimes x = 0 \in I \otimes H$$

For $n \geq 1$ result follows by (iii)

$\square$

(v) By (iii)

The antipode S

Given a bialgebra H.

An <u>antipode</u> for H is a K-module hom

$$S : H \to H$$

such that for all $a \in H$,

$$\sum_{(a)} a_1 S(a_2) = \varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2$$

where we recall

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2$$