

Lec 5 Friday Sept 16

Given a k -coalgebra C

Recall a coideal I of C is a

k -submodule I of C st

$$\Delta(I) \subseteq C \otimes I + I \otimes C,$$

$$\varepsilon(I) = 0$$

Ex Recall tensor alg $T(V) \cong C$ is coalg.

Take $J =$ 2-s. ideal ideal gen by $uv-vu$ $u, v \in V$

then J is a coideal of $T(V)$

pf For $x \in J$ show $\Delta(x) \in C \otimes J + J \otimes C$

wlog $x = \gamma (uv-vu) z$ $\gamma, z \in C$ $u, v \in V$

$$\begin{aligned} \Delta(x) &= \Delta(\gamma) \underbrace{\Delta(uv-vu)}_{\parallel} \Delta(z) \\ &= \Delta(\gamma) (uv-vu \otimes 1 + 1 \otimes (uv-vu)) \Delta(z) \\ &\subseteq C \otimes J + J \otimes C \end{aligned}$$

Also show

$$\begin{aligned} \varepsilon(x) &= 0 \\ \varepsilon(x) &= \varepsilon(\gamma) \underbrace{\varepsilon(uv-vu)}_{\parallel} \varepsilon(z) \\ &= 0 \end{aligned}$$



Graded algebras and coalgebras

Given K -module V

A grading of V is a sequence $\{V_n\}_{n \in \mathbb{N}}$ of K -submodules whose direct sum is V

Possibly $V_n = 0$ for some n

Ex. Take $V = K$

$$K_0 = K$$
$$K_n = 0 \quad n = 1, 2, \dots$$

Given K -modules U, V Consider K -module $U \otimes V$.

Given gradings $\{U_n\}_{n \in \mathbb{N}}$ of U
 $\{V_n\}_{n \in \mathbb{N}}$ of V

For $n \in \mathbb{N}$ define

$$(U \otimes V)_n = U_0 \otimes V_n + U_1 \otimes V_{n-1} + \dots + U_n \otimes V_0$$
$$\subseteq U \otimes V$$

Then

$$\{ (U \otimes V)_n \}_{n \in \mathbb{N}}$$

is a grading of $U \otimes V$
— o —

Given k -algebra A

Given grading $\{A_n\}_{n \in \mathbb{N}}$ of A

Call this an algebra grading whenever

$$A_i A_j \subseteq A_{i+j} \quad i, j \in \mathbb{N}$$

$$1_A \in A_0$$

LEM For above A TFAE

(i) $\{A_n\}_{n \in \mathbb{N}}$ is an algebra grading

(ii) $\forall n \in \mathbb{N}$

$$m \left((A \otimes A)_n \right) \subseteq A_n$$

$$u \left(\kappa_n \right) \subseteq A_n$$

pf (i) \rightarrow (ii) $\forall n \quad 0 \leq i \leq n$

$$m(A_i \otimes A_{n-i}) = A_i A_{n-i} \subseteq A_n$$

Also

$$u(k_0) = u(1) \subseteq k \mathbb{1}_A \subseteq A_0$$

and for ≥ 1

$$u(k_n) = u(0) = 0 \subseteq A_n$$

(iii) \rightarrow (i)

$$\forall n \quad i, j \in \mathbb{N}$$

$$A_i A_j = m(A_i \otimes A_j) \subseteq m((A \otimes A)_{i+j}) \subseteq A_{i+j}$$

$$\mathbb{1}_A = u(1) \in u(k_0) \subseteq A_0$$

□

The above lemma suggests how to define a coalgebra grading.

Given coalgebra C

Given grading $\{C_n\}_{n \in \mathbb{N}}$ of C

Call this a coalgebra grading whenever

$$\Delta(C_n) \subseteq (C \otimes C)_n \quad n \in \mathbb{N}$$

$$\varepsilon(C_n) \subseteq K_n$$

LEM With above notation TFAE

(i) $\{C_n\}_{n \in \mathbb{N}}$ is a coalgebra grading

(ii) $F_n \quad n \in \mathbb{N}$

$$\Delta(C_n) \subseteq C_0 \otimes C_n + C_1 \otimes C_{n-1} + \dots + C_n \otimes C_0$$

$$\varepsilon(C_n) = 0 \quad \text{if } n \geq 1.$$

pf

Routine

□

Given a bialgebra H

Given a grading $\{H_n\}_{n \in \mathbb{N}}$ of H

Call this a bialgebra grading whenever

it is both an algebra and coalgebra grading.

Ex Given K -module V

(i) For the tensor algebra $T(V)$ the grading

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

is a bialgebra grading

(ii) For the symmetric algebra $S(V)$ the grading

$$S(V) = \sum_{n \in \mathbb{N}} \text{Sym}^n(V)$$

is a bialgebra grading

LEM Given graded bialgebra $H = \bigoplus_{n \in \mathbb{N}} H_n$

Then for the set P of primitive elements

$$(i) \quad P = \bigoplus_{n \in \mathbb{N}} (P \cap H_n)$$

(ii) P is a coideal of H

pf (i) $\forall x \in P$

write

$$x = \sum_{n \in \mathbb{N}} x_n$$

$x_n \in H_n$

$\forall n \in \mathbb{N}$ show

$$x_n \in P$$

We have

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

So

$$0 = x \otimes 1 + 1 \otimes x - \Delta(x)$$

$$= \sum_{n \in \mathbb{N}} \underbrace{(x_n \otimes 1 + 1 \otimes x_n - \Delta(x_n))}_{\in (H \otimes H)_n}$$

$\in (H \otimes H)_n$

So each summand is 0

$$\Delta(x_n) = x_n \otimes 1 + 1 \otimes x_n$$

$$\text{so } x_n \in P$$

(ii) show

$$\Delta(P) \subseteq P \otimes H + H \otimes P,$$

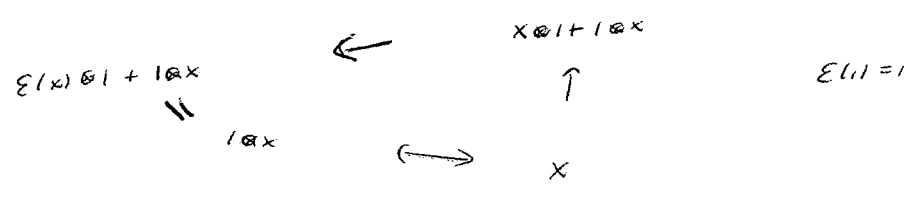
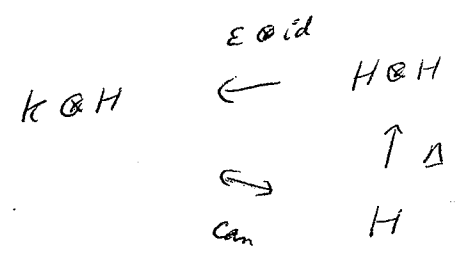
$$\varepsilon(P) = 0.$$

For $x \in P$

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$\in P \otimes H$ $\in H \otimes P$

Also using



So

$$\varepsilon(x) = 0$$



Given graded algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$

obs A_0 is a \mathbb{k} -subalgebra of A ,

$$u(\mathbb{k}) \subseteq A_0$$

Call the grading connected whenever

$$u(\mathbb{k}) = A_0$$

LEM Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

(i) The maps

$$\eta: K \rightarrow H_0$$

$$\epsilon|_{H_0}: H_0 \rightarrow K$$

are inverses and hence bijections

(ii)

$$\ker(\epsilon) = \underbrace{\bigoplus_{n \geq 1} H_n}_{\text{is def } I}$$

(iii) For $n \geq 1$ and $x \in H_n$

$$\Delta(x) = x \otimes 1 + 1 \otimes x \in \bigoplus_{i=1}^n H_i \otimes H_{n-i}$$

(iv) For $x \in H$

$$\Delta(x) = x \otimes 1 \in H \otimes I$$

$$\Delta(x) = 1 \otimes x \in I \otimes H$$

(v) For $x \in I$

$$\Delta(x) = x \otimes 1 + 1 \otimes x \in I \otimes I$$

pf (i) Since u, ε are K -linear and
 $\varepsilon(1_H) = 1, \quad u(1) = 1_H$

(ii) \supseteq : $\varepsilon(H_n) = 0$ for $n \geq 1$ since coalg H
 is graded

\subseteq For $x \in \ker(\varepsilon)$ show $x \in I$

write $x = \sum_{n \in \mathbb{N}} x_n \quad x_n \in H_n$

$$0 = \varepsilon(x) = \sum_{n \in \mathbb{N}} \varepsilon(x_n)$$

$$\varepsilon(x_n) = 0 \text{ for } n \geq 1$$

$$= \varepsilon(x_0)$$

Now $x_0 = 0$ by (i)

So $x = \sum_{i=1}^n x_i \in I$

(iii) Since $\text{coalg } H$ is graded

$$\begin{aligned} \Delta(H_n) &\subseteq \bigoplus_{i=0}^n H_i \otimes H_{n-i} \\ &= \underbrace{H_0 \otimes H_n}_{1_H \otimes H_n} + \underbrace{H_n \otimes H_0}_{H_n \otimes 1_H} + \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i} \end{aligned}$$

So

$$\Delta(x) = 1 \otimes \varphi(x) + \varphi(x) \otimes 1 \in \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i} \quad (*)$$

show $\varphi(x) = x = \phi(x)$

Using

$$x = \sum_{(x)} \varepsilon(x_1) x_2$$

(*) gives

$$x = \underbrace{\varepsilon(1)}_1 \varphi(x) + \underbrace{\varepsilon(\phi(x))}_0 1 \in \bigoplus_{i=1}^{n-1} \underbrace{\varepsilon(H_i)}_0 \otimes H_{n-i}$$

So $x = \varphi(x)$

Using $x = \sum_{(x)} x_1 \varepsilon(x_2)$

(*) yields $x = \phi(x)$

(iv) WLOG $x \in H_n$

$$F_n = 0$$

$$\Delta(x) - x \otimes 1 = 0 \in H \otimes I$$

$$\Delta(x) - 1 \otimes x = 0 \in I \otimes H$$

$F_n \neq 1$ result follows by (iii)

(v) By (iii)

□

The antipode S

Given a bialgebra H .

An antipode for H is a K -module hom

$$S: H \rightarrow H$$

such that for all $a \in H$,

$$\sum_{(a)} a_1 S(a_2) = \varepsilon(a) 1_H = \sum_{(a)} S(a_1) a_2$$

where we recall

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2$$