Lyndon Words & Shuffles 12/12/16

<u>Intro</u>: Today I'm going to introduce a combinatorial object called <u>Lyndon words</u>. On their surface, they don't seem particularly relevant to the study of Hopf algebras, but they actually form an algebraically independent generating set for a couple different important Hopf algebras. For example, a shuffle algebra over a field of characteristic 0 can be viewed as a polynomial algebra over the Lyndon words.

<u>Def'n</u>: Fix a totally ordered set $\mathcal{A}$ called the <u>alphabet</u>.

- A <u>word</u> over $\mathcal{A}$ is a finite tuple of elts of $\mathcal{A}$. Let $\mathcal{A}^*$ denote the set of words/ $\mathcal{A}$.
- Let $\varnothing$ denote the empty word.
- For a word $w \in \mathcal{A}^*$, let w_i denote the ith letter of w, ie, the ith entry of the tuple w.
- Let $\ell(w)$ denote the number of letters in w, ie, the length of the word w.
- The <u>concatenation</u> of two words u, v is the word $u_1 \ldots u_{\ell(u)} v_1 \ldots v_{\ell(v)}$, written uv.
- A <u>prefix</u> of $w \in \mathcal{A}^*$ is a word $u \in \mathcal{A}^*$ st $\exists\, v \in \mathcal{A}^*$ st $w = uv$.

- A <u>suffix</u> of $w \in \mathcal{A}^*$ is a word $v \in \mathcal{A}^*$ st $\exists u \in \mathcal{A}^*$ st $w = uv$. v is a <u>proper suffix</u> if $u \neq \emptyset$.

- We define a relation $\leq$ on the set $\mathcal{A}^*$ as follows: For $u, v \in \mathcal{A}^*$, $u \leq v$ iff
  - either $\exists\, i \in \{1, \dots, \min\{\ell(u), \ell(v)\}\}$ st $u_i < v_i$ & $\forall j \in \{1, \dots, i-1\}$, $u_j = v_j$, or
  - the word u is a prefix of v.

<u>Fact:</u> $\leq$ totally orders $\mathcal{A}^*$. This can be proven with simple case analysis. $\leq$ is called the <u>lexicographic order</u> on $\mathcal{A}^*$.

<u>Ex:</u> $113 \leq 114$, $113 \leq 132$, $19 \leq 195$, $41 \leq 421$, $539 \leq 54$,
$\emptyset \leq w \ \forall w \in \mathcal{A}^*$.

<u>Rmk:</u> $\leq$ does not respect concatenation from the right. For example, if $u = 1$, $v = 13$, $w = 4$, then $u \leq v$ but $uw > uv$.

<u>Prop 6.2</u> Let $a, b, c, d \in \mathcal{A}^*$.

b) If $c \leq d$, then $ac \leq ad$.

c) If $ac \leq ad$, then $c \leq d$.

d) If $a \leq c$, then $ab \leq cd$ or a is a prefix of c.

e) If $ab \leq cd$, then $a \leq c$ or c is a prefix of a.

f) If $ab \leq cd$ & $\ell(a) \leq \ell(c)$, then $a \leq c$.

g) If $a \leq b \leq ac$, then a is prefix of b.

h) If a is a prefix of b, $a \leq b$.

i) If a and b are prefixes of c, then a is a prefix of b or vice versa.

j) If $a \leq b$ & $\ell(a) \geq \ell(b)$, then $ac \leq bc$.

k) If $b \neq \emptyset$, $a < ab$.

Proof of prop 6.2: Case analysis. Excluded.

Qn: When do words commute?

Prop 6.4: Let $u, v \in \mathcal{A}^*$ satisfy $uv = vu$. Then $\exists\, t \in \mathcal{A}^*$ and $n, m \in \mathbb{N}$ st $u = t^n$ & $v = t^m$.

Pf: Induction on $\ell(u) + \ell(v)$. Assume $\ell(u)$ & $\ell(v) > 0$. Observe that either u is a prefix of v or vice versa, ie, the shorter word must be a prefix of the longer one. WLOG assume u a prefix of v. Then $\exists\, w \in \mathcal{A}^*$ st $v = uw$. We have $\ell(u) + \ell(w) = \ell(v)$. Also, $vu = uv \Rightarrow uwu = uuw \Rightarrow wu = uw$. By induction, $\exists\, t \in \mathcal{A}^*$ $\exists\, n, m \in \mathbb{N}$ st $u = t^n$ & $w = t^m$. Then $v = uw = t^n t^m = t^{n+m}$, so we're done. $\square$

(*) Prop 6.5: Let $u, v, w \in \mathcal{A}^*$ be nonempty words st $uv \geq vu$, $vw \geq wv$, & $wu \geq uw$. Then $\exists\, t \in \mathcal{A}^*$ & $n, m, p \in \mathbb{N}$ st $u = t^n$, $v = t^m$, $w = t^p$.

Pf: Induction on $\ell(u) + \ell(v) + \ell(w)$. This proof is slightly more complicated than the proof of 6.4, but it is not radically different. It uses 6.2 heavily. For these reasons, it is omitted. $\square$

(*) Cor 6.6: Let $u, v, w \in \mathcal{A}^*$ satisfy $uv \geq vu$ & $vw \geq wv$. If $v \neq \emptyset$, then $uw \geq wu$.

Pf: Assume the contrary, so $uw < wu$, so $wu \geq uw$. Assume $u, w \neq \emptyset$ (else, this is clear). By 6.5,

$\exists\, t \in \mathcal{A}^*$ & $n, m, p \in \mathbb{N}$ s.t. $u = t^n$, $v = t^m$, $w = t^p$.
But then $wu = t^p t^n = t^{p+n} = t^n t^p = uw$, contradicting $uw < wu$. Thus $uw \geq wu$, as required. $\square$.

Now, we define Lyndon words.

<u>Defn:</u> A word $w \in \mathcal{A}^*$ is <u>Lyndon</u> if it is nonempty & $\forall$ nonempty proper suffix v of w, $v > w$.

<u>Ex:</u> 113 is Lyndon: its nonempty proper suffixes are 13 & 3, & $113 < 13 < 3$.

2424 is not Lyndon: 24 is a non-$\emptyset$ proper suffix of 2424, but $24 < 2424$.

<u>Ex:</u>
- Every word of length 1 is Lyndon.
- A word $w = w_1 w_2$ of length 2 is Lyndon iff $w_1 < w_2$.
- A word $w = w_1 w_2 w_3$ is Lyndon iff $w_1 < w_3$ & $w_1 \leq w_2$.
- A word $w = w_1 w_2 w_3 w_4$ is Lyndon iff $w_1 < w_4$, $w_1 \leq w_3$, $w_1 \leq w_2$, & if $w_1 = w_3$, then $w_2 < w_4$.

These rules get more complicated as the length of the word increases.

Now we'll talk about some properties of Lyndon words.

Prop 6.14: Let w be a Lyndon word & u, v two words st $w = uv$.

a) If v is nonempty, then $v \geq w$.

b) If v is nonempty, then $v > u$.

c) If $u, v \neq \emptyset$, then $vu > uv$.

d) $vu \geq uv$.

Pf: Clear.

Cor 6.15: Let w be a Lyndon words & v a nonempty suffix of w. Then $v \geq w$.

Pf: 6.14 a).

Prop 6.16: Let u, v be Lyndon words st $u < v$.

Then a) The word uv is Lyndon.

b) We have $uv < v$.

Pf: b) u is Lyndon, so it's nonempty. Thus $uv \neq v$.
Assume $uv \not< v\emptyset$. Since $u < v$, Prop 6.2 d) implies $uv \leq v\emptyset$ or u is a prefix of v, so by assumption, u is a prefix of v.
Thus $\exists$ $t \in \mathfrak{a}^*$ st $v = ut$. Such a t is nonempty, else $u \not< v$. So t is a nonempty proper suffix of v, so $t > v$. Prop 6.2 b) $\Rightarrow$ $uv \leq ut = v$. Since $uv \neq v$, $uv < v$. This completes pf of b).

a) $v \neq \emptyset$ since Lyndon. So $uv \neq \emptyset$. Need to check every non-$\emptyset$ proper suffix of uv is $> uv$. Let p be non-$\emptyset$ proper suffix of uv. There are two cases:

1) p is a non-$\emptyset$ suffix of v.

2) p has the form qv where q is a non-$\emptyset$ proper suffix of u.

First, handle case 1. Since v is Lyndon, $p \geq v$, & by part b), $uv < v$, so $p \geq v > uv$. So $p > uv$.

Case 2: $p = qv$ where q is a non-$\emptyset$ proper suffix of u. We have $q > u$. By 6.2d), either $uv \leq qv$ or u is a prefix of q. Since u is not a prefix of q, $uv \leq qv$. $uv \neq qv$ (else $u = q$), we have $uv < qv = p$. case done.

So $p > uv$ always, so uv is Lyndon. $\square$.

Cor 6.17: Let u & v be two Lyndon words s.t. $u < v$. Let z be a word s.t. $zv \geq vz$ & $uz \geq zu$. Then $z = \emptyset$.

Pf: Assume not. Then Cor 6.6 $\Rightarrow uv \geq vu$. By (6.16b), $uv < v \leq vu$, which is a contradiction. $\square$.

Prop 6.18: Let u & v be Lyndon. Then $u \geq v \iff uv \geq vu$.

Pf: There are 3 cases: 1) $u < v$, 2) $u = v$, 3) $u > v$.

Case 1: $u < v \Rightarrow uv < v$ (by 6.16b) $\leq vu$.

Case 2: Then $u = v$ & $uv = vu$.

Case 3: $u > v \Rightarrow vu < u \leq uv$ by prop (6.16b) again.

We now define an important feature of Lyndon words: A bijection between all words & multisets of Lyndon words. This is vital for constructing polynomial gen. sets for Shuffle algs.

<u>Def'n</u>: Let w be a word. A Chen-Fox-Lyndon factorization (CFL factorization) of that word is a tuple $(a_1, \ldots, a_k)$ of Lyndon words s.t. $w = a_1 \cdots a_k$, & $a_1 \geq a_2 \geq \cdots \geq a_k$.

<u>Example</u>: $(23, 2, 14, 13323, 13, 12, 12, 1)$ is a CFL factorization of 23214133231312121.

<u>Thm 6.27</u>: Let w be a word. Then $\exists!$ CFL factorization of w.

<u>Lemma 6.28</u>: Let $(a_1, a_2, \ldots, a_k)$ be a CFL factorization of w. If p is a nonempty suffix of w, then $p \geq a_k$.

<u>Pf</u>: Induction on k.

<u>Base</u>: If $k=1$, then w is Lyndon, so $p \geq a_1$.

<u>Induction Step</u>: Let K be a positive integer. Assume 6.28 for $k=K$. NTS it holds for $k=K+1$.

Let $(a_1, \ldots, a_{K+1})$ be a CFL factorization of w. Let p be a nonempty suffix of w. We NTS $p \geq a_{K+1}$. We have $w = a_1 \cdots a_{K+1}$ w/ $a_1 \geq \cdots \geq a_{K+1}$. Let $w' = a_2 \cdots a_{K+1}$. Then $w = a_1 w'$. Every nonempty suffix of w is either a nonempty suffix of w' or has the form qw' for a nonempty suffix q of a_1. Thus there are 2 cases: 1) $p =$ non-$\emptyset$ suffix of w', or

2) $p = qw'$.

Consider case 1. Since $w' = a_2 \cdots a_{k+1}$, $(a_2, \ldots, a_{k+1})$ is a CFL factorization of w' of length k, so by induction, $p \geq a_{k+1}$. So case 1 is done.

Consider Now case 2, $p = qw'$. Since a_1 is Lyndon, $q \geq a_1$; so we have $q \geq a_1 \geq \cdots \geq a_{k+1}$. Thus $p = qw' \geq q \geq a_{k+1}$. ∎.

Pf of Thm 6.27: First, we'll prove ∃ a CFL factorization of w. It is clear there exists a tuple of Lyndon words $(a_1, \ldots, a_k)$ st $w = a_1 \cdots a_k$ [let each word be 1 letter long]. Fix such a tuple w/ minimum k. Then I claim $a_1 \geq \cdots \geq a_k$. If $a_i < a_{i+1}$, then (by Prop 6.16a) $a_i a_{i+1}$ is Lyndon. Thus $(a_1, \ldots, a_i a_{i+1}, \ldots a_k)$ would also be a tuple of Lyndon words with $w = a_1 \cdots (a_i a_{i+1}) \cdots a_k$ having length $k-1$, so k was not the minimum length of such a tuple. Thus all $a_i \geq a_{i+1}$, & so a CFL factorization exists.

Now, we'll show there is at most one such factorization. Suppose $(a_1, \ldots, a_k)$ & $(b_1, \ldots, b_m)$ are two CFL factorizations of w. $w = b_1 \cdots b_m$, so b_m is a non-∅ suffix of w. By Lemma 6.28, $b_m \geq a_k$. Likewise, $a_k \geq b_m$, so $a_k = b_m$. Continue like this to obtain $a_i = b_i \; \forall i$, so that the CFL factorization of w is unique. ∎

(*) Now we examine a factorization of Lyndon words into smaller words, called the <u>standard factorization</u>. It will be useful for performing induction over Lyndon words.

(*) <u>Thm 6.30</u>: Let w be a Lyndon word of length >1. Let v be the lexicographically smallest nonempty proper suffix of w. Since v is proper, $\exists\, u \neq \emptyset$ in $\mathfrak{A}^*$ st $w = uv$. Consider this u. Then:

a) The words u & v are Lyndon;

b) We have $u < w < v$.

<u>Pf</u>: a) Every nonempty proper suffix of $v \geq v$ (since every such suffix is also a suffix of w, & v was lexicographically smallest). Hence $> v$ since proper. v is nonempty, so v is Lyndon.

b) Since w is Lyndon, every NPS of w is $> w$; here $w < v$. v is nonempty, so $u < uv = w < v$.

a) If p is a non-$\emptyset$ ps of u, then pv is NPS of w, so $pv > w = uv$. Prop 6.2 e) gives either $u \leq p$ or p is a prefix of u. Assume TAC $p \leq u$. Then p is a prefix of u, ie, $\exists\, q \in \mathfrak{A}^*$ st $u = pq$. We have $w = uv = pqv = p(qv)$, so qv is a proper suffix of w. Further, $q \neq \emptyset$, so qv is a NPS of w. By assumption, $v \leq qv$. By prop 6.2b), $pv \leq pqv$.

So $pv \le pqv = w$, contradicting $pv > w$.
Thus, it is false that $p \le u$, so $p > u$. p was arbitrary, so <u>u is Lyndon</u>. $\square$.

We now want to connect the theory of Lyndon words w/ the notion of shuffle products. Recall:

<u>Defn</u>: a) Let $n \in \mathbb{N}$, $m \in \mathbb{N}$. Then $Sh_{n,m}$ denotes the subset $\{\sigma \in \mathfrak{S}_{n+m} : \sigma^{-1}(1) < \dots < \sigma^{-1}(n),\ \sigma^{-1}(n+1) < \dots < \sigma^{-1}(n+m)\}$ of $\mathfrak{S}_{n+m}$.

b) Let $u = (u_1, \dots, u_n)$ & $v = (v_1, \dots, v_m)$ be two words. If $\sigma \in Sh_{n,m}$, then $u \underset{\sigma}{\sqcup\!\sqcup} v$ will denote the word $(w_{\sigma(1)}, \dots, w_{\sigma(n+m)})$, where $(w_1, \dots, w_{n+m})$ is the concatenation $uv = (u_1, \dots, u_n, v_1, \dots, v_m)$. Note that the multiset of all the letters of $u \underset{\sigma}{\sqcup\!\sqcup} v$ is the disjoint union of the multiset of all letters of u with the multiset of all letters from v; hence, $\ell(u \underset{\sigma}{\sqcup\!\sqcup} v) = \ell(u) + \ell(v)$.

c) Let $u = (u_1, \dots, u_n)$ & $v = (v_1, \dots, v_m)$ be two words. The <u>multiset of shuffles of u & v</u> is the multiset $\{(w_{\sigma(1)}, \dots, w_{\sigma(n+m)}) \mid \sigma \in Sh_{n,m}\} = \{u \underset{\sigma}{\sqcup\!\sqcup} v \mid \sigma \in Sh_{n,m}\}$. It is denoted $u \sqcup\!\sqcup v$.

I will state the main result relating shuffle algebras & Lyndon words:

Thm 6.71 (Radford): Assume $\mathbb{Q}$ is a subring of k. Let V be a free k-module with basis $(b_a)_{a \in \mathfrak{A}}$, where $\mathfrak{A}$ is a totally ordered set. Then the shuffle algebra $Sh(V)$ is a polynomial k-algebra. An algebraically independent generating set of $Sh(V)$ can be constructed as follows:

For every word $w \in \mathfrak{A}^*$ over the alphabet $\mathfrak{A}$, let us define an elt b_w of $Sh(V)$ by $b_w = b_{w_1} b_{w_2} \cdots b_{w_\ell}$, where ℓ is the length of w. [The multiplication here is in $T(V)$]. Let $\mathfrak{L}$ denote the set of all Lyndon words over $\mathfrak{A}$. Then $(b_w)_{w \in \mathfrak{L}}$ is an algebraically independent generating set of the k-algebra $Sh(V)$.

Ex: elements of $Sh(V)$ written as polynomials in this generating set:

- $b_{12} = b_{12}$ (since 12 is Lyndon);
- $b_{21} = b_1 \sqcup\!\sqcup\, b_2 - b_{12}$ [$\sqcup\!\sqcup$ is multiplication in $Sh(V)$].
- $b_{11} = \frac{1}{2} b_1 \sqcup\!\sqcup\, b_1$
- $b_{213} = b_2 \sqcup\!\sqcup\, b_{13} - b_{123} - b_{132}$
- $b_{321} = b_1 \sqcup\!\sqcup\, b_2 \sqcup\!\sqcup\, b_3 - b_{23} \sqcup\!\sqcup\, b_1 - b_3 \sqcup\!\sqcup\, b_{12} + b_{132}$

Before we have the tools to prove this theorem, we'll need the following fact, which provides the main connection between Lyndon words & shuffles:

Thm 6.44: Let u & v be two words. Let $(a_1, \dots, a_p)$ be the CFL factorization of u, & $(b_1, \dots, b_q)$ be the CFL factorization of v.

a) Let $(c_1, \dots, c_{p+q})$ be the result of sorting the list $(a_1, \dots, a_p, b_1, \dots, b_q)$ in decreasing order. Then the lexicographically highest elt of $u \sqcup\!\sqcup v$ is $c_1 \cdots c_{p+q}$ (& $(c_1, \dots, c_{p+q})$ is the CFL factorization of this elt).

b) Let $\mathcal{L}$ denote the set of all Lyndon words. If w is a Lyndon word & z is any word, let $\text{mult}_w z$ denote the # of terms in the CFL factorization of z which are equal to w. The multiplicity with which the lexicographically highest elt of the multiset $u \sqcup\!\sqcup v$ appears in the multiset $u \sqcup\!\sqcup v$ is $\prod_{w \in \mathcal{L}} \binom{\text{mult}_w u + \text{mult}_w v}{\text{mult}_w u}$ (the product is well-defined b/c almost all its terms are 1).

c) If $a_i \geq b_j \ \forall i \in \{1, \dots, p\}, j \in \{1, \dots, q\}$, then the lex highest elt of the multiset $u \sqcup\!\sqcup v$ is uv.

d) If $a_i > b_j \ \forall i \in \{1, \dots, p\}, j \in \{1, \dots, q\}$, then the multiplicity with which the word uv appears in $u \sqcup\!\sqcup v$ is 1.

e) Assume u is Lyndon. & $u \geq b_j \ \forall j \in \{1, \dots, q\}$. Then the lex highest elt of the multiset $u \sqcup\!\sqcup v$ is uv, & the multiplicity with which this word appears in $u \sqcup\!\sqcup v$ is $\text{mult}_u v + 1$.

Ex: Let $u = 23232$, $v = 323221$ over $\mathcal{A} = \{1,2,3,\ldots\}$.
The CFL factorizations of u & v are $(23,23,2)$ & $(3,23,2,2,1)$ respectively. In the notation of Thm 6.44 a), $p=3$, $q=5$. 6.44a) predicts that the lex-highest elt of $u \sqcup\!\sqcup v$ is $c_1 \cdots c_8$ where
$(c_1, \ldots, c_8) = (3,23,23,23,2,2,2,1)$.
6.44 b) predicts that 32323232221 appears in $u \sqcup\!\sqcup v$ with a multiplicity of $\prod_{w \in \mathcal{L}} \binom{\mathrm{mult}_w u + \mathrm{mult}_w v}{\mathrm{mult}_w u}$.
All but finitely many factors are 1: the only factors which are not 1 are those corresponding to Lyndon words w which appear in both the CFL factorizations of u & v, since for any other factor, at least one of $\mathrm{mult}_w u$ or $\mathrm{mult}_w v = 0$, & so $\binom{\mathrm{mult}_w u + \mathrm{mult}_w v}{\mathrm{mult}_w u} = 1$.
In our example, the only factors which are not 1 are those for $w = 23$ & $w = 2$. So:

$$\prod_{w \in \mathcal{L}} \binom{\mathrm{mult}_w u + \mathrm{mult}_w v}{\mathrm{mult}_w u} = \binom{\mathrm{mult}_{23} u + \mathrm{mult}_{23} v}{\mathrm{mult}_{23} u}\binom{\mathrm{mult}_2 u + \mathrm{mult}_2 v}{\mathrm{mult}_2 u}$$
$$= \underbrace{\binom{2+1}{2}}_{=3} \cdot \underbrace{\binom{1+2}{1}}_{=3}$$
$$= \boxed{9}.$$

In order to prove Thm 6.44, we'll need to prove some stronger statements, for which we need some new notation.

Defn: a) If $p, q \in \mathbb{Z}$, then $\underline{[p:q]^+}$ denotes $\{p+1, \dots, q\}$. Obs $|[p:q]^+| = q-p$ if $q \geq p$.

b) If I, J are non-$\emptyset$ intervals of $\mathbb{Z}$, then $\underline{I < J}$ iff every $i \in I$ & $j \in J$ satisfy $i < j$.

c) If w is a word with n letters, I an interval of $\mathbb{Z}$ st $I \subset [0:n]^+$, then $\underline{w[I]}$ denotes the word $(w_{p+1}, \dots, w_q)$ where $I = [p:q]^+$, $q \geq p$. A word of the form $w[I]$ is called a <u>factor</u> of w.

d) Let α be a composition. Define a tuple <u>intsys α</u> of intervals of $\mathbb{Z}$ as follows: Write $\alpha = (\alpha_1, \dots, \alpha_\ell)$ where $\ell = \ell(\alpha)$. Then, set

intsys $\alpha = (I_1, \dots, I_\ell)$, where

$$I_i = \left[\sum_{k=1}^{i-1} \alpha_k : \sum_{k=1}^{i} \alpha_k\right]^+ \quad \forall i \in \{1, \dots, \ell\}.$$

Then intsys α is an ℓ-tuple of non-$\emptyset$ intervals of $\mathbb{Z}$. This tuple is called the <u>interval system</u> corresponding to α.

Ex: Let $\alpha = (4,1,4,2,3)$. Then the interval system corresponding to α is:

intsys $\alpha = ([0:4]^+, [4:5]^+, [5:9]^+, [9:11]^+, [11:14]^+)$

$= (\{1,2,3,4\}, \{5\}, \{6,7,8,9\}, \{10,11\}, \{12,13,14\})$.

Rmk: a) If $I, J \neq \emptyset$ are intervals of $\mathbb{Z}$ st $I < J$, then I & J are disjoint.

b) If I & J are disjoint non-$\emptyset$ intervals of $\mathbb{Z}$, then $I < J$ or $J < I$.

c) Let α be a composition. Write $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ where $\ell = \ell(\alpha)$. Then intsysα is the ! ℓ-tuple $(I_1, \ldots, I_\ell)$ of non-$\emptyset$ intervals of $\mathbb{Z}$ st

1) $I_1, \ldots, I_\ell$ form a set partition of $[0:n]^+$, $n = |\alpha|$;

2) $I_1 < \ldots < I_\ell$;

3) $|I_i| = \alpha_i \ \forall i \in \{1, \ldots, \ell\}$.

Pf: Easy.

The following lemmas are consequences of the definitions of elts of $Sh_{n,m}$ & shuffle algs:

Lemma 6.50: Let $n \in \mathbb{N}$ & $m \in \mathbb{N}$. Let $\sigma \in Sh_{n,m}$.

a) If I is an interval of $\mathbb{Z}$ st $I \subseteq [0:n+m]^+$, then $\sigma(I) \cap [0:n]^+$ and $\sigma(I) \cap [n:n+m]^+$ are intervals.

b) Let L, K be non-$\emptyset$ intervals of $\mathbb{Z}$ st $K \subset [0:n]^+$ & $L \subseteq [0:n]^+$ & $K < L$ & st $K \cup L$ is an interval. Assume $\sigma^{-1}(K)$ & $\sigma^{-1}(L)$ are intervals, but $\sigma^{-1}(K) \cup \sigma^{-1}(L)$ is not. Then $\exists P \neq \emptyset$ st $P \subset [n:n+m]^+$ & $\sigma^{-1}(P)$, $\sigma^{-1}(K) \cup \sigma^{-1}(P)$, & $\sigma^{-1}(P) \cup \sigma^{-1}(L)$ are intervals & st $\sigma^{-1}(K) < \sigma^{-1}(P) < \sigma^{-1}(L)$.

c) Lemma 6.50 b) remains valid if "$K \subset [0:n]^+$, $L \subset [0:n]^+$," & "$P \subset [n:n+m]^+$" are replaced by "$K \subset [n:n+m]^+$ & $L \subset [n:n+m]^+$" & "$P \subset [0:n]^+$", respectively.

Lemma 6.52: Let u & v be two words. Let $n = \ell(u)$, $m = \ell(v)$; $\sigma \in Sh_{n,m}$.

a) If I is an interval satisfying either $I \subset [0:n]^+$ or $I \subset [n:n+m]^+$, & $\sigma^{-1}(I)$ is an interval, then

(6.4) $$(u \underset{\sigma}{\sqcup\!\sqcup} v)[\sigma^{-1}(I)] = (uv)[I].$$

b) Assume that $u \underset{\sigma}{\sqcup\!\sqcup} v$ is the lex highest elt of $u \sqcup\!\sqcup v$.

Let $I \subset [0:n]^+$ & $J \subset [n:n+m]^+$ be two non-$\emptyset$ intervals. Assume $\sigma^{-1}(I)$ & $\sigma^{-1}(J)$ are also intervals, that $\sigma^{-1}(I) < \sigma^{-1}(J)$, & $\sigma^{-1}(I) \cup \sigma^{-1}(J)$ is an interval as well. Then $(uv)[I] \cdot (uv)[J] \geq (uv)[J] \cdot (uv)[I]$.

c) 6.52 b) valid if "$I \subset [0:n]^+$ & $J \subset [n:n+m]^+$" is replaced by "$I \subset [n:n+m]^+$ & $J \subset [0:n]^+$".

Pf: Exercise.

If time go to Defn 6.54,

Ex 6.55, ...