§4.9. Hall Algebra $\to$ (quotient alg)

Hopf subalgebra of $A(GL)$ related to unipotent conjugacy classes in $GL_n(\mathbb{F}_q)$

<u>Defn</u>: $g \in GL_n(\mathbb{F}_q)$ is <u>unipotent</u> if $g - id$ is nilpotent.

A conjugacy class in $GL_n(\mathbb{F}_q)$ is <u>unipotent</u> if its elements are <u>unipotent</u>.

$\mathcal{H}_n :=$ subspace of $R_{\mathbb{C}}(GL_n)$ whose supports is on unipotent conjugacy classes

$= \{ f \in R_{\mathbb{C}}(GL_n) : \operatorname{supp}(f) \subseteq \{\text{unipotent conjugacy classes}\} \}$.

$\mathcal{H} = \bigoplus_n \mathcal{H}_n$

<u>Prop</u>: $\mathcal{H}$ is graded connected self-dual w.r.t $(\ ,\ )$ <u>Hopf subalgebra</u> of $A_{\mathbb{C}}(GL)$

$\mathcal{H}$ is a quotient Hopf algebra of $A_{\mathbb{C}}(GL)$ as $A_{\mathbb{C}}(GL) \twoheadrightarrow \mathcal{H}$ is <u>Hopf algebra homomorphism</u>

So kernel $=: \mathcal{H}^{\perp}$ is both an ideal and two-sided coideal.

<u>Pf</u>: $$(\chi_i \cdot \chi_j)(g) = \frac{1}{|P_{ij}|} \sum_{\substack{h \in GL_n \\ h^{-1}gh = \begin{bmatrix} g_i & * \\ 0 & g_j \end{bmatrix}}} \chi_i(g_i)\chi_j(g_j).$$

g is unipotent $\Leftrightarrow$ g_i, g_j are unipotents.

$$\Delta(\chi)(g_i, g_j) = \frac{1}{|q^{ij}|} \sum_{g = \begin{bmatrix} g_i & * \\ 0 & g_j \end{bmatrix}} \chi(g). \quad \Rightarrow \quad \Delta(\mathcal{H}) \subseteq \mathcal{H} \otimes \mathcal{H}$$

<u>Defn</u>: $\mathcal{H} :=$ Hall algebra. $\quad \lambda \in \mathrm{Par}_n$

$J_\lambda :=$ GL_n-conjugacy class of unipotent matrices of Jordan type λ.

$z_\lambda(q) := \# |C(J_\lambda)|$

$\quad$ centralizer of an element of J_λ

The ~~indication~~ indicator class function $\{\underline{1}_{J_\lambda}\}$ forms a $\mathbb{C}$-basis for $\mathcal{H}$ whose multiplicative structure constants are called Hall coefficients $g^{\lambda}_{\mu,\nu}(q)$:
$$\underline{1}_{J_\mu} \underline{1}_{J_\nu} = \sum_{\lambda} g^{\lambda}_{\mu,\nu}(q)\, \underline{1}_{J_\lambda}$$

Since $(\underline{1}_{J_\lambda}, \underline{1}_{J_\lambda}) = \frac{|J_\lambda|}{|G|} = \frac{1}{|z_\lambda(q)|}$ so $\underline{1}_{J_\lambda}$, $z_\lambda(q)\underline{1}_{J_\lambda}$ are dual basis.

$\mathcal{H}$ is self-dual $\Rightarrow$
$$\Delta(\underline{1}_{J_\lambda}) = \sum_{\mu,\nu} g^{\lambda}_{\mu,\nu}(q) \frac{z_\mu(q) z_\nu(q)}{z_\lambda(q)} \underline{1}_{J_\mu} \otimes \underline{1}_{J_\nu}$$

$g^{\lambda}_{\mu,\nu}(q)$ has the following geometric interpretation:

If $g \in GL_n(\mathbb{F}_q)$ is in J_λ, then $g^{\lambda}_{\mu,\nu}(q)$ counts the g-stable $\mathbb{F}_q$-subspaces $V \subset \mathbb{F}_q^n$ for which $g|_V$ is in J_μ and the induced map $\bar{g}$ on $\mathbb{F}_q^n/V$ is in in J_ν

<u>examples</u>: If $\lambda = (3,2,1)$ so $g = \begin{pmatrix} 1&1&&&&\\ &1&1&&&\\ &&1&&&\\ &&&1&1&\\ &&&&1&\\ &&&&&1 \end{pmatrix}$ and $\mu = (1,1,1)$ then

$$g^{\lambda}_{\mu,\nu}(q) = \begin{cases} 1 & \text{if } \nu = (2,1) \\ 0 & \text{o.w.} \end{cases}$$

Look at

<u>Pf</u>: $\mathbb{1}_{J_\mu} \circ \mathbb{1}_{J_\nu} = \frac{1}{|P_{i,j}|} \cdot \left|\left\{ h \in GL_n : \underbrace{h^{-1} g h}_{\text{change of basis operation.}} = \begin{bmatrix} g_i & * \\ 0 & g_j \end{bmatrix} \text{ with } g_i \in J_\mu,\ g_j \in J_\nu \right\}\right|$. #

<u>Next</u>: Show $\varphi: \Lambda_{\mathbb{C}} \xrightarrow{\cong} A(GL)(i)_{\mathbb{C}} \hookrightarrow A(GL)_{\mathbb{C}} \twoheadrightarrow \mathcal{H}$ is a Hopf algebra isomorphism

<u>Thm</u>: φ is isom sending

$$h_n \longmapsto \sum_{\lambda \in Par_n} \mathbb{1}_{J_\lambda} \quad (1)$$

$$e_n \longmapsto q^{\binom{n}{2}} \mathbb{1}_{J_{(1^n)}} \quad (2)$$

$$P_n \longmapsto \sum_{\lambda \in Par_n} (q;q)_{\ell(\lambda)-1} \mathbb{1}_{J_\lambda} \quad (3).$$

<u>Pf</u>. It is enough to check the formula for P_n to conclude isomorphism.

<u>Why?</u>, φ being an graded Hopf algebra homomorphism is ~~automati~~ automatic.

since $\dim_{\mathbb{C}} \Lambda_n = |Par_n| = \dim_{\mathbb{C}} H_n$, we only need to check it is injective.

Now $\Lambda_{\mathbb{C}} \cong \mathbb{C}[P_1, \cdots, P_n, \cdots]$ as $\mathbb{C}$-algebras.

and $\deg(p_i) = i$. So we only need to verify $\varphi(p_i) \neq 0$ $\forall i$

• $h_n \longmapsto \sum_{\lambda\in Par_n} \underline{1}_{J_\lambda}$ is automatic since $\Lambda_{\mathbb{C}} \xrightarrow{\cong} A(GL)(i)$

$h_n \longmapsto \underline{1}_{GL_n}$

Now we want to deduce ② and ③ from ① using relations between h_n, e_n and p_n.

Remember $H(t) = \sum_{n\geq 0} h_n t^n \quad E(t) = \sum_{n\geq 0} e_n t^n \quad P(t) = \sum_{n\geq 0} p_{n+1} t^n$

and we have $H(t)E(-t)=1$ and $H'(t)E(-t) = P(t)$

$H(t) = \prod_i (1+x_i t) \quad E(t) = \prod_i \frac{1}{1-x_i t} \qquad P(t) = \sum_i \frac{x_i}{1+x_i t}$

Let $\varphi[[t]] : \Lambda[[t]] \longrightarrow \mathcal{H}[[t]]$ be the induced map from φ is an algebra homomorphism

and let $\tilde{h}_n := \sum_{\lambda\in Par_n} \underline{1}_{J_\lambda} \quad \tilde{e}_n := q^{\binom{n}{2}} \underline{1}_{J_{(1^n)}} \quad \tilde{p}_n := \sum_{\lambda \in Par_n} (q;q)_{\ell(\lambda)-1} \underline{1}_{J_\lambda}.$

and $\tilde{H}(t) = \sum_{n\geq 0} \tilde{h}_n t^n \quad \tilde{E}(t) = \sum_{n\geq 0} \tilde{e}_n t^n \quad \tilde{P}(t) = \sum_{n\geq 0} \tilde{p}_{n+1} t^n$

Since $\varphi[[t]](H(t)) = \tilde{H}(t)$, If we can show $\tilde{H}(t)\tilde{E}(-t) = 1$ & $\tilde{H}'(t)\tilde{E}(-t) = \tilde{P}(t)$ (*)

we have $\cancel{\varphi(h_n) = \tilde{h}_n}$ $\varphi(e_n) = \tilde{e}_n$ & $\varphi(p_n) = \tilde{p}_n$

(*) is equivalent to $\sum_{k=0}^n (-1)^k \tilde{e}_k \tilde{h}_{n-k} = 0$ and $\sum_{k=0}^n (-1)^k \tilde{e}_k (n-k) \tilde{h}_{n-k} = \tilde{p}_n$

To compute $\tilde{e}_k \cdot \tilde{h}_{n-k}$, we compute $\underline{1}_{J_{(1^k)}} \cdot \underline{1}_{J_\lambda}(g)$, $g \in J_\mu$, $\ell = \ell(\mu)$

Suppose g has ℓ Jordan blocks, by previous proposition

$$\underline{1}_{J_{(1^k)}} \cdot \underline{1}_{J_\lambda}(g) = \sum_\mu g^{\mu}_{(1^k),\lambda}(q) \cdot \underline{1}_{J_\mu}$$

where $g^\mu_{(1^k),\lambda}(q) = \# \left| Grass_{\mathbb{F}_q}(k, \ell) \right| = \frac{(q;q)_\ell}{(q;q)_k (q;q)_{\ell-k}} = \frac{\frac{(q;q)_\ell}{(1-q)^\ell}}{\frac{(q;q)_k}{(1-q)^k} \cdot \frac{(q;q)_{\ell-k}}{(1-q)^{\ell-k}}}$

$= \begin{bmatrix} \ell \\ k \end{bmatrix}_q$

# of completes flags in $\mathbb{F}_q^\ell$

# of complete flags in $\mathbb{F}_q^k$

# of complete flags in $\mathbb{F}_q^{\ell-k}$

Rmk: $\# \left| \mathbb{P}^n_{\mathbb{F}_q} \right| = \frac{q^{n+1}-1}{q-1}$

so $\sum_{k=0}^n (-1)^k \tilde{e}_k \tilde{h}_{n-k} = \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \sum_{\lambda\in Par_n} \cancel{(q;q)_{\ell(\lambda)-1}} \underline{1}_{J_{(1^k)}} \cdot \underline{1}_{J_\lambda}.$

$= \cancel{\sum_{k=0}^n \sum_{\lambda\in Par_n} (-1)^k (q;q)_{\ell(\lambda)-1} \sum_{\mu} q^{\binom{n}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q} = \sum_{\lambda\in Par_n} \cancel{(q;q)_{\ell(\lambda)-1}} \sum_{k=0}^n \sum_\mu (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \ell(\mu) \\ k \end{bmatrix}_q \underline{1}_{J_\mu}.$

$= \sum_{\lambda\in Par_n} \cancel{(q;q)_{\ell(\lambda)-1}} \sum_\mu \sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q \underline{1}_{J_\mu} = 0$

$$\sum_{k=0}^{n} (-1)^k (n-k)\, \tilde{e}_k \tilde{h}_{n-k} = \sum_{k=0}^{n} (-1)^k (n-k) \sum_{\lambda \in \mathrm{Par}_n} q^{\binom{k}{2}} \underline{1}_{J_{(1^k)}} \underline{1}_{J_\lambda}$$

$$= \sum_{\lambda \in \mathrm{Par}_n} \sum_{\mu} \underbrace{\sum_{k=0}^{l} (-1)^k (n-l+l-k)\, q^{\binom{k}{2}} \begin{bmatrix} l \\ k \end{bmatrix}_q}_{=0} \underline{1}_{J_\mu} = 0 \qquad \#$$

<u>Next</u>, State Hall's Theorem showing that the classical Hall algebra and $\mathcal{H}$ are isom.

Classical Hall's algebras has $\mathbb{Z}$-basis elements $\{u_\lambda\}_{\lambda \in \mathrm{Par}}$ with

$$u_\mu u_\nu = \sum_\lambda g^\lambda_{\mu,\nu}(p)\, u_\lambda$$

where $g^\lambda_{\mu,\nu}(p) := \# \left| \left\{ \underset{\text{SES}}{0 \to M \to L \to N \to 0} : L \cong \bigoplus_{i=1}^{l(\lambda)} \mathbb{Z}/p^{\lambda_i}\mathbb{Z} \text{ and } M \cong \bigoplus_{i=1}^{l(\mu)} \mathbb{Z}/p^{\mu_i}\mathbb{Z} \right\} \right|$.

If $q = p$ is prime.

The map $u_\lambda \longmapsto \underline{1}_{J_\lambda}$ identifies classical one and $\mathcal{H}_{\mathbb{Z}}$ as $\mathbb{Z}$-algebras.

(Hall's Theorem), (R, m) is a DVR (PID) w/ unique maximal ideal m

Residue field : $\#|k = R/m| = q$. $\quad L$: R-mod of type λ, where $N = L/M$ is of type ν

M: R-mod of type μ.

\# such $M \subseteq L$ is given by. $\left[g^\lambda_{\mu,\nu}(t) \right]_{t=q}$. where $g^\lambda_{\mu,\nu}(t)$ is a polynomial in $\mathbb{Z}[t]$ called. Hall polynomial. independent of (R, m)

Furthermore, $\deg g^\lambda_{\mu,\nu}(t) \leq n(\lambda) - (n(\mu) + n(\nu))$.

"$<$" means $g^\lambda_{\mu,\nu} = 0$, coefficient of $t^{n(\lambda)-(n(\mu)+n(\nu))}$ is $C^\lambda_{\mu,\nu}$

<u>example</u>; Take $R = \mathbb{F}_p[[t]]$ and $m = (t)$ then $k = \mathbb{F}_p$.

and $g^\lambda_{\mu,\nu}(p)$ is the same as multiplicative structure constant in $\mathcal{H}$

Take $R = \mathbb{Z}_p$ and $m = (p)$ then $g^\lambda_{\mu,\nu}(p)$ is the ~~mos~~ same as structure constant in classical Hall algebra. $R/m = \mathbb{Z}/p\mathbb{Z}$.

So the map is isomorphic. #

Review:

$$(\mathrm{Ind}_H^G f)(g) = \frac{1}{|H|} \sum_{\substack{k \in G \\ kgk^{-1} \in H}} f(kgk^{-1})$$

V^K : K-fixed space of $\mathbb{C}[G]$-module, $K \triangleleft G$

$$\chi_{V^K}(\underbrace{gK}_{G/K}) = \frac{1}{|K|} \sum_{h \in gK} \chi_V(h)$$

q-binomial theorem

$$\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q x^{\ell-k} = (x-1)(x-q) \cdots (x-q^{\ell-1}).$$

so plug in $x=1$ we have.

$$\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q = 0$$

$$\begin{bmatrix} \ell \\ k \end{bmatrix}_q = \frac{(q;q)_\ell}{(q;q)_k (q;q)_{\ell-k}}.$$

and $(x;q)_m := (1-x)(1-qx) \cdots (1-q^{m-1}x) \quad \forall m \in \mathbb{N}$

$\frac{d}{dx}$ both side & plug in $x=1$, we have.

$$\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} (\ell-k) \begin{bmatrix} \ell \\ k \end{bmatrix}_q = (1-q) \cdots (1-q^{\ell-1}) = (q;q)_{\ell-1}$$