Lec 4 Wednesday Sept 14

Examples of bialgebras

Ex Given k-module V

Recall tensor alg $T(V)$ is both alg and coalg.

By constr Δ, ε are k-alg morphisms.

So $T(V)$ is bialg.

Similarly $S(V)$ is bialg.

Ex Given group G.

Recall Group alg kG is alg + coalg with

$m(t_g \otimes t_h) = t_{gh}$ $\qquad g, h \in G$

$u(1) = t_e$ $\qquad e =$ ident of G

$\Delta(t_g) = t_g \otimes t_g$

$\varepsilon(t_g) = 1$

Check if kG is bialg

$a, b \in kG$

WLOG $a = t_g$, $b = t_h$, $g, h \in G$

$$\Delta(ab) \overset{?}{=} \Delta(a)\,\Delta(b)$$

$$\Delta(ab) = \Delta(t_{gh}) = t_{gh} \otimes t_{gh}$$

$$\Delta(a)\,\Delta(b) = \underbrace{(t_g \otimes t_g)(t_h \otimes t_h)}_{} = t_g t_h \otimes t_g t_h = t_{gh} \otimes t_{gh}$$

OK

$$\varepsilon(ab) \overset{?}{=} \varepsilon(a)\,\varepsilon(b)$$

$$\varepsilon(ab) = \varepsilon(t_{gh}) = 1$$

$$\varepsilon(a)\,\varepsilon(b) = \varepsilon(t_g)\,\varepsilon(t_h) = 1 \cdot 1$$

OK

$A = kG$

$$\Delta(1_A) \overset{?}{=} 1_A \otimes 1_A$$

$$\Delta(1_A) = \Delta(t_e) = t_e \otimes t_e$$

$$1_A \otimes 1_A = t_e \otimes t_e$$

ok

$$\varepsilon(1_A) \overset{?}{=} 1$$

$$\varepsilon(1_A) = \varepsilon(t_e)$$

OK.

kG is bialg $\square$

LEM Given coalgebras C, C', D, D'

and coalg morphisms

$\varphi : C \to D, \qquad \varphi' : C' \to D'$

then the map

$$\varphi \otimes \varphi' : C \otimes C' \to D \otimes D'$$
$$c \otimes c' \to \varphi(c) \otimes \varphi'(c')$$

is a coalg morphism.

pf Given φ is coalg morph so

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "\varphi \otimes \varphi"'] & D \otimes D \end{tikzcd}$$

$\forall c \in C$

$$\begin{tikzcd} c \arrow[r] \arrow[d] & \varphi(c) \arrow[d] \\ \sum_{(c)} c_1 \otimes c_2 \arrow[r] & \sum_{(c)} \varphi(c_1) \otimes \varphi(c_2) \end{tikzcd}$$

$$\sum_{(\varphi(c))} (\varphi(c))_1 \otimes (\varphi(c))_2$$

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\varepsilon_C"'] & D \arrow[d, "\varepsilon_D"] \\ K \arrow[r, "id"'] & K \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r] \arrow[d] & \varphi(c) \arrow[d] \\ \varepsilon(c) \arrow[r, phantom, "="] & \varepsilon(\varphi(c)) \end{tikzcd}$$

Similar for φ'

Require

$$
\begin{tikzcd}
C\otimes C' \arrow[r, "\varphi\otimes\varphi'"] \arrow[d, "\Delta_{C\otimes C'}"'] & D\otimes D' \arrow[d, "\Delta_{D\otimes D'}"] \\
C\otimes C'\otimes C\otimes C' \arrow[r] & D\otimes D'\otimes D\otimes D'
\end{tikzcd}
$$

We have

$$
\begin{tikzcd}
C\otimes C' \arrow[r, "\varphi\otimes\varphi'"] \arrow[d, "\Delta_C\otimes\Delta_{C'}"'] \arrow[dd, bend right=60, "\Delta_{C\otimes C'}"'] \arrow[dr, phantom, "\mathrm{Com.}"] & D\otimes D' \arrow[d, "\Delta_D\otimes\Delta_{D'}"] \arrow[dd, bend left=60, "\Delta_{D\otimes D'}"] \\
C\otimes C\otimes C'\otimes C' \arrow[r, "\varphi\otimes\varphi\otimes\varphi'\otimes\varphi'"'] \arrow[d, "1\otimes\tau\otimes1"] \arrow[dr, phantom, "\mathrm{Com}"] & D\otimes D\otimes D'\otimes D' \arrow[d, "1\otimes\tau\otimes1"] \\
C\otimes C'\otimes C\otimes C' \arrow[r, "\varphi\otimes\varphi'\otimes\varphi\otimes\varphi'"'] & D\otimes D'\otimes D\otimes D'
\end{tikzcd}
$$

ok

Require

$$\begin{array}{ccc} C \otimes C' & \xrightarrow{\varphi \otimes \varphi'} & D \otimes D' \\ \varepsilon_{C\otimes C'} \downarrow & & \downarrow \varepsilon_{D \otimes D'} \\ k & \xrightarrow[id]{} & k \end{array}$$

$\forall\, c \in C,\ c' \in C'$ chase $c \otimes c'$ around diag

$$\begin{array}{ccc} c \otimes c' & \longrightarrow & \varphi(c) \otimes \varphi'(c') \\ \downarrow & & \downarrow \\ \varepsilon(c)\varepsilon'(c') & = & \underbrace{\varepsilon(\varphi(c))}_{\varepsilon(c)} \; \underbrace{\varepsilon'(\varphi'(c'))}_{\varepsilon(c')} \end{array}$$

ok

$\square$

For a K-algebra A

recall K-module hom

$$u : K \to A$$
$$1 \to 1_A$$

u might not be injective

Now suppose A is a bialgebra.

then the composition

$$K \xrightarrow{u} A \xrightarrow{\varepsilon} K$$
$$1 \to 1_A \to 1$$

is the identity map on K.

So u is injective.

So for $\alpha \in K$

$$\alpha 1_A = 0 \quad \text{iff} \quad \alpha = 0$$

Given bialg A

For $a \in A$, call a <u>primitive</u> whenever

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

[so for $A = T(V)$, each element of V is primitive]

Define P = set of primitive elements in A

LEM For above P:

(i) P is a K-submodule of A

(ii) For $a, b \in P$

$ab - ba \in P$

(iii) For $a \in P$, $\varepsilon(a) = 0$

pf (i) For $\alpha \in K$ and $a \in P$ show $\alpha a \in P$:

$$\Delta(\alpha a) \overset{?}{=} (\alpha a) \otimes 1 + 1 \otimes (\alpha a)$$

$$\|$$

$$\alpha \Delta(a) \qquad \text{ok}$$

$$\|$$

$$\alpha(a \otimes 1 + 1 \otimes a)$$

(ii) observe

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$= (a\otimes 1 + 1\otimes a)(b\otimes 1 + 1\otimes b)$$

$$= ab\otimes 1 + a\otimes b + b\otimes a + 1\otimes ab$$

Also

$$\Delta(ba) = ba\otimes 1 + a\otimes b + b\otimes a + 1\otimes ba$$

So

$$\Delta(ab - ba) = (ab-ba)\otimes 1 + 1\otimes(ab-ba)$$

so $ab - ba \in P$

(iii) Since $\Delta(a) = a\otimes 1 + 1\otimes a$

and $a = \sum_{(a)} \varepsilon(a_1)\, a_2$

so $a = \varepsilon(a)\, 1_A + \underset{\overset{\shortparallel}{1}}{\varepsilon(1)}\, a$

so $\varepsilon(a)\, 1_A = 0$

so $\varepsilon(a) = 0$ $\square$

Given coalg C

For $c \in C$, call c <u>grouplike</u> whenever both

$$\Delta(c) = c \otimes c,$$
$$\varepsilon(c) = 1$$

let G = set of all grouplike elements in C

LEM Assume C is a bialgebra. Then for $x, y \in G$

$$xy \in G$$

pf
$$\Delta(xy) = \Delta(x)\Delta(y)$$
$$= x \otimes x \quad y \otimes y$$
$$= xy \otimes xy$$

$$\varepsilon(xy) = \underbrace{\varepsilon(x)}_{=1}\underbrace{\varepsilon(y)}_{=1}$$
$$= 1$$

$\square$

More on tensor algebras

Given k-module V

Recall for bialg $T(V)$

$$\Delta(a) = a \otimes 1 + 1 \otimes a \qquad a \in V$$

$$\Delta(ab) = ab \otimes 1 + a \otimes b + b \otimes a + 1 \otimes ab \qquad b \in V$$

$\forall c \in C$

$\Delta(abc) =$

	$\otimes$
abc	1
bc	a
ac	b
ab	c
c	ab
b	ac
a	bc
1	abc

one term for each subset of $\{1, 2, 3\}$

For $n \geq 0$ and

$$a_1, a_2, \ldots, a_n \in V$$

describe

$$\Delta(a_1 a_2 \cdots a_n)$$

Consider set $\Omega = \{1, 2, \ldots, n\}$

For a subset $S \subseteq \Omega$ $\quad s = |S|$

write

$$S = \{ i_1, i_2, \dots, i_s \} \qquad i_1 < i_2 < \cdots < i_s$$

Define

$$a_S = a_{i_1} a_{i_2} \cdots a_{i_s} \in V^{\otimes s}$$

Then

$$\Delta(a_1 a_2 \cdots a_n) = \sum_{S \subseteq \Omega} a_S \otimes a_{\overline{S}}$$

So

$$\Delta(V^{\otimes n}) \subseteq \sum_{s=0}^{n} V^{\otimes s} \otimes V^{\otimes (n-s)}$$

Also note

$$\varepsilon(V^{\otimes n}) = 0 \qquad \text{if } n \geq 1$$

Given coalg C

Given k-submodule $J \leq C$

Consider quotient k-module C/J

When does C/J inherit a coalg str from C?

Recall

$$\text{can}: \begin{array}{l} C \to C/J \\ c \mapsto c+J \end{array} \quad \text{is } k\text{-module hom}$$

The map

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\text{can} \otimes \text{can}} C/J \otimes C/J \qquad (*)$$

is k-module hom

Desire $J \leq \ker(*)$

this holds if

$$\Delta(J) \subseteq C \otimes J + J \otimes C$$

In this case $(*)$ induces a k-module hom

$$\tilde{\Delta} : C/J \longrightarrow C/J \otimes C/J$$

By constr

$$\begin{tikzcd} C \arrow[r, "\text{can}"] \arrow[d, "\Delta"'] & C/J \arrow[d, "\tilde{\Delta}"] \\ C \otimes C \arrow[r, "\text{can} \otimes \text{can}"'] & C/J \otimes C/J \end{tikzcd}$$

Consider k-module hom

$$\varepsilon : C \longrightarrow k$$

Desire $\varepsilon(J) = 0$

In this case ε induces a k-module hom

$$\tilde{\varepsilon} : C/J \longrightarrow k$$

By constr

$$\begin{tikzcd} C \arrow[r, "\text{can}"] \arrow[d, "\varepsilon"'] & C/J \arrow[d, "\tilde{\varepsilon}"] \\ k \arrow[r, "\text{id}"'] & k \end{tikzcd}$$

Call J a <u>coideal of C</u>

whenever both

$$\Delta(J) \subseteq C \otimes J + J \otimes C$$

$$\varepsilon(J) = 0$$

LEM Assume J is a coideal of C

then C/J is a coalg with coproduct

$\tilde{\Delta}$ and counit $\tilde{\varepsilon}$.

Moreover

$$\text{can}: C \longrightarrow C/J$$

is a coalg morphism.

pf. One checks $\tilde{\Delta}, \tilde{\varepsilon}$ make their diagrams commute.

the map can: $C \to C/J$ is a coalg morphism

by the discussion above the lemma.

$\square$