Notation

Throughout, we fix the following notation:

q is a fixed prime power

$GL_n = GL_n(\mathbb{F}_q)$

$A = A(GL) = \bigoplus_{n\geq 0} R(GL_n)$

$\Sigma = \bigsqcup_{n\geq 0} \mathrm{Irr}(GL_n)$, PSH basis of A

$\mathfrak{P} \subset A$ set of primitive elements

$\mathcal{C} = \Sigma \cap \mathfrak{P}$.

We embed $GL_i \times GL_j \hookrightarrow GL_{i+j}$ block-diagonally: $\begin{pmatrix} GL_i & 0 \\ 0 & GL_j \end{pmatrix}$.

$P_{i,j} = \left\{ \begin{pmatrix} GL_i & * \\ 0 & GL_j \end{pmatrix} \right\} \subset GL_{i+j}$ (called a parabolic subgroup)

$K_{i,j} = \left\{ \begin{pmatrix} I_i & * \\ 0 & I_j \end{pmatrix} \right\}$, so we have exact $1 \to K_{i,j} \to P_{i,j} \to GL_i \times GL_j \to 1$

We identify $R(GL_i) \otimes R(GL_j) \approx R(GL_i \times GL_j)$ via the isomorphism defined by $\varphi \otimes \psi \mapsto \varphi * \psi$, where $(\varphi * \psi)(g,h) = \varphi(g) \cdot \psi(h)$.

Define $\mathrm{res}_{i,j} : R(GL_{i+j}) \to R(GL_i) \otimes R(GL_j)$

by $\mathrm{res}_{i,j}\varphi = \left[\mathrm{Res}^{GL_{i+j}}_{P_{i,j}} \varphi \right]^{K_{i,j}}$

$\mathrm{ind}_{i,j} : R(GL_i) \otimes R(GL_j) \to R(GL_{i+j})$

by $\mathrm{ind}_{i,j}(\varphi \otimes \psi) = \mathrm{Ind}^{GL_{i+j}}_{P_{i,j}} \mathrm{Inf}^{P_{i,j}}_{GL_i \times GL_j} (\varphi \otimes \psi)$

The bialgebra structure on A is given by

$\varphi \cdot \psi = \mathrm{ind}_{i,j}(\varphi \otimes \psi) \qquad (\varphi \in A_i,\ \psi \in A_j)$

$\Delta \varphi = \sum_{i+j=n} \mathrm{res}_{i,j} \varphi \qquad (\varphi \in A_n)$.

General Linear Groups

Let $A = A(GL) = \bigoplus_{n\geq 0} R(GL_n)$. We have seen that A is a PSH,

with PSH basis $\Sigma = \bigsqcup_{n\geq 0} \mathrm{Irr}(GL_n)$ consisting of irreducible characters.

As usual, $\#$ is primitives in A, and we put $\mathcal{C} = \Sigma \cap \#$.

By Thm 3.12, we have a decomposition:

$$A = \bigotimes_{\rho\in\mathcal{C}}' A(\rho)$$

Here, $\otimes'$ denotes a "restricted tensor product": It is the group of formal symbols $\bigotimes_{\rho\in\mathcal{C}} a_\rho$, where all but finitely many a_ρ are 1, subject to the usual $\mathbb{Z}$-multilinear relations.

Alternatively, we may take $\bigotimes_{\rho\in\mathcal{C}}' A(\rho) = \varinjlim_{\substack{F\subset\mathcal{C} \\ |F|<\infty}} \bigotimes_F A(\rho)$,

the direct limit being over all finite subsets of $\mathcal{C}$.

$\bigotimes_{\rho\in\mathcal{C}}' A(\rho)$ comes with a graded bialgebra structure, and our decomposition is an iso of such.

(It is worth it to note that $\bigotimes_{\rho\in\mathcal{C}}' A(\rho)$ is the coproduct of the set $\{A(\rho)\}_{\rho\in\mathcal{C}}$ in the category of $\mathbb{Z}$-algebras)

Def: $\mathcal{C}_n = \mathcal{C} \cap \mathrm{Irr}(GL_n)$ is the set of "cuspidal" representations of GL_n
$= \# \cap \mathrm{Irr}(GL_n)$. For $\rho\in\mathcal{C}$, we write $d(\rho) = n$ if $\rho\in\mathcal{C}_n$.

Fact: A rep V of GL_n is cuspidal iff $V^{K_{i,j}} = 0 \ \forall\, i+j=n,\ i,j>0$

PF: $$\mathrm{res}(\chi_V) = \chi_V \otimes 1 + 1 \otimes \chi_V + \sum_{\substack{i+j=n \\ i,j>0}} \mathrm{res}_{i,j}(\chi_V)$$

So χ_V primitive iff $\mathrm{res}_{i,j}(\chi_V) = \chi_{V^{K_{i,j}}}$ is zero $\forall i,j$.

QED.

We want to count $|\mathcal{C}_n|$ (one important feature here is that $|\mathcal{C}_n| \neq 0 \ \forall n$! This starkly contrasts with the case of symmetric groups/wreath products).

To do this, we first need to count $|\mathrm{Irr}(GL_n)|$.

For each $n \geq 1$, let $\mathcal{F}_n = \{ f \in \mathbb{F}_q[x] \mid f \text{ is monic irreducible, } \deg f = n,\ f \neq x \}$.
$\mathcal{F} = \bigcup_{n\geq 1} \mathcal{F}_n$

Prop: $|\mathrm{Irr}(GL_n)| = \#(\text{conjugacy classes of } GL_n)$ ~~$= |\{\text{functions } \lambda : \bigcup_{i\geq 1} \mathbb{F}_q \to \mathrm{Par}\}|$~~

$$= \left| \left\{ \text{functions } \lambda : \mathcal{F} \to \mathrm{Par} \ \middle|\ \sum_{f\in\mathcal{F}} (\deg f) \cdot |\lambda(f)| = n \right\} \right|$$

PF: The first equality is a general fact about representations of finite groups.

The second follows from the observation that conj. classes of GL_n are in bijection with rational canonical forms in GL_n.

Equivalently, conj. classes are in bijection with $\mathbb{F}_q[x]$-module structures on $\mathbb{F}_q^n$ in which x is invertible (up to isomorphism).

But by the structure thm for PID's, these are in bijection with

$\{ \lambda : \mathcal{F} \to \mathrm{Par} \mid \sum_{f\in\mathcal{F}} (\deg f) |\lambda(f)| = n \}$ (we exclude x from $\mathcal{F}$ so that x will not annihilate anything in $\mathbb{F}_q^n$).

QED.

Prop (4.46): $|\mathcal{C}_n| = |\mathcal{F}_n|$.

Pf: We do induction on n. When $n=1$, noting that all $\chi \in A_1$ are primitive, we have $|\mathcal{C}_1| = |\mathrm{Irr}(GL_1)| = |\mathrm{Irr}(\mathbb{F}_q^\times)|$.

But $\mathbb{F}_q^\times$ is abelian, so $|\mathrm{Irr}(\mathbb{F}_q^\times)| = |\mathbb{F}_q^\times| = |\mathcal{F}_1|$.

Now let $n>1$. This is where we use the PSH structure on A.

Recall the notation $\mathbb{N}_{fin}^{\mathcal{C}} = \{ f : \mathcal{C} \to \mathbb{N} \mid f \text{ has finite support} \}$.

Since the iso $A \approx \bigotimes'_{\rho \in \mathcal{C}} A(\rho)$ respects gradings, we have

$$A_n = \bigoplus_{\substack{\alpha \in \mathbb{N}_{fin}^{\mathcal{C}} \\ \sum_{\mathcal{C}} d(\rho)\alpha_\rho = n}} \bigotimes_{\rho \in \mathcal{C}} A(\rho)_{\alpha_\rho}$$

Here $A(\rho)_{\alpha_\rho}$ is the α_ρ-graded part in grading such that $\underline{\deg \rho = 1}$.

But $A(\rho) \approx \Lambda$, so $A(\rho)_{\alpha_\rho}$ has basis parametrized by $\mathrm{Par}(\alpha_\rho)$.

Hence, comparing bases on both sides of the above equality,

$$|\mathrm{Irr}(GL_n)| = \left| \left\{ \text{functions } \lambda : \mathcal{C} \to \mathrm{Par} \;\middle|\; \sum_{\mathcal{C}} d(\rho) \cdot |\lambda(\rho)| = n \right\} \right|$$

This gives

$$|\mathcal{C}_n| = |\mathrm{Irr}(GL_n)| - \left| \left\{ \lambda : \bigsqcup_{1}^{n-1} \mathcal{C}_i \to \mathrm{Par} \;\middle|\; \sum_{\mathcal{C}} d(\rho)\, |\lambda(\rho)| = n \right\} \right|$$

Similarly, last prop gives

$$|\mathcal{F}_n| = |\mathrm{Irr}(GL_n)| - \left| \left\{ \lambda : \bigsqcup_{1}^{n-1} \mathcal{F}_i \to \mathrm{Par} \;\middle|\; \sum_{\mathcal{F}} \deg(f) \cdot |\lambda(f)| = n \right\} \right|.$$

By induction hypothesis, $|\mathcal{C}_i| = |\mathcal{F}_i|$ for $i<n$, so our prop follows.

QED.

Now how do we write down $|\mathcal{F}_n|$?

As shown above, $|\mathcal{C}_1| = |\mathcal{F}_1| = q-1$.

When $n>1$, $\mathcal{F}_n$ is simply all monic irreducibles in $\mathbb{F}_q[x]$ of degree n.

By elementary field theory, $q^n = |\mathbb{F}_{q^n}| = \sum_{d|n} d \cdot I_d$.

where $I_d = |\mathcal{F}_d|$, unless $d=1$, when $I_d = q$, and the sum is over all positive divisors of n.

But then the Möbius inversion formula says

$$n I_n = \sum_{d|n} q^d \mu\left(\tfrac{n}{d}\right),$$

where $\mu(m) = \begin{cases} 0 & \text{if } m \text{ not square free} \\ (-1)^{\#(\text{prime factors of } m)} & \text{if } m \text{ is square free} \end{cases}$,

Thus:

<u>Prop:</u> $|\mathcal{C}_1| = |\mathcal{F}_1| = q-1$, and for $n \geq 2$, $|\mathcal{C}_n| = |\mathcal{F}_n| = \frac{1}{n} \sum_{d|n} q^d \mu\left(\frac{n}{d}\right)$.

(Note that for n <u>prime</u>, this is simple: $|\mathcal{C}_n| = |\mathcal{F}_n| = \frac{1}{n}(q^n - q)$.)

<u>Example:</u> Take $q=2$. Then have the following table for the number of cuspidal representations of $GL_n(\mathbb{F}_2)$:

n	1	2	3	4	5	6	7	8	9	10
$\lvert\mathcal{C}_n\rvert$	1	1	2	3	6	9	18	30	56	99

<u>Cor:</u> $|\mathcal{C}_n| \sim \frac{q^n}{n}$ as $n \to \infty$. (or as $q \to \infty$)

<u>Pf:</u> Since the number of divisors of n is at most n, the last prop gives

$$\frac{1}{n}\left(q^n - n q^{n/2}\right) \leq |\mathcal{C}_n| \leq \frac{1}{n}\left(q^n + n q^{n/2}\right) \quad \forall n \geq 2,$$

hence $$1 - n q^{-n/2} \leq \frac{|\mathcal{C}_n|}{q^n/n} \leq 1 + n q^{-n/2}.$$

QED.

Unipotent Characters

We will need to use exercise 4.31(d) here:

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$. We put $G_\alpha = GL_{\alpha_1} \times \cdots \times GL_{\alpha_m}$.

If $n = \sum \alpha_i$, then we have embedding $G_\alpha \hookrightarrow GL_n$ as $\begin{pmatrix} GL_{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & GL_{\alpha_m} \end{pmatrix}$.

We also define the parabolic subgroup $P_\alpha = \left\{ \begin{pmatrix} GL_{\alpha_1} & & * \\ & \ddots & \\ 0 & & GL_{\alpha_m} \end{pmatrix} \right\} \subset GL_n$.

We have natural projection $P_\alpha \to G_\alpha$.

Then we define $\operatorname{ind}^n_\alpha : R(GL_{\alpha_1}) \otimes \cdots \otimes R(GL_{\alpha_m}) \longrightarrow R(GL_n)$

by $\operatorname{ind}^n_\alpha(\varphi_1 \otimes \cdots \otimes \varphi_m) = \operatorname{Ind}^{GL_n}_{P_\alpha} \operatorname{Inf}^{P_\alpha}_{GL_\alpha} (\varphi_1 \otimes \cdots \otimes \varphi_m)$

Exercise 4.31(d): $\operatorname{ind}^n_\alpha(\varphi_1 \otimes \cdots \otimes \varphi_m) = \varphi_1 \cdots \varphi_m$, where the right is multiplication in A.

(One way to prove this is to note that the left is adjoint to the map $\operatorname{res}^n_\alpha : R(GL_n) \to R(GL_{\alpha_1}) \otimes \cdots \otimes R(GL_{\alpha_m})$, (see the same exercise for def of $\operatorname{res}^n_\alpha$) while the right is $\operatorname{ind}_{\alpha_1, \alpha_2 + \cdots + \alpha_m} \circ (1 \otimes \operatorname{ind}_{\alpha_2, \alpha_3 + \cdots + \alpha_m}) \circ \cdots \circ (1 \otimes \cdots \otimes 1 \otimes \operatorname{ind}_{\alpha_{m-1}, \alpha_m})$

which is adjoint to $(1 \otimes \cdots \otimes 1 \otimes \operatorname{res}_{\alpha_{m-1}, \alpha_m}) \circ \cdots \circ (1 \otimes \operatorname{res}_{\alpha_2, \alpha_3 + \cdots + \alpha_m}) \circ \operatorname{res}_{\alpha_1, \alpha_2 + \cdots + \alpha_m}$.

By an argument used to show A is coassociative, one shows

$$\operatorname{res}^n_\alpha = (1 \otimes \cdots \otimes 1 \otimes \operatorname{res}_{\alpha_{m-1}, \alpha_m}) \circ \cdots \circ \operatorname{res}_{\alpha_1, \alpha_2 + \cdots + \alpha_m}$$
)

For each n let 1_{GL_n} denote the trivial representation of GL_n. Note that $1_{GL_1} = 1_{\mathbb{F}_q^\times}$ is cuspidal, so that $A(1_{GL_1})$ appears as a factor in our decomposition $A = \bigotimes'_{\rho \in \mathcal{C}} A(\rho)$.

Def: The unipotent characters of GL_n are precisely those $\chi \in \mathrm{Irr}(GL_n)$ such that $\langle \chi, (1_{GL_1})^n \rangle \neq 0$; i.e., they are the elements of the PSH basis of $A(1_{GL_1})$ in degree n.

Fact: (1) $(1_{GL_1})^n$ is the character of the representation $\mathbb{C}[GL_n/B]$ of GL_n, where B is the subgroup of upper triangular matrices.

(2) The unipotent representations of GL_n are the irreducible subrepresentations of $\mathbb{C}[GL_n/B]$. Equivalently, they are the irreducible representations V of GL_n such that $V^B \neq 0$.

Pf: (1) By exercise 4.31(d), $(1_{GL_1})^n = \mathrm{Ind}_B^{GL_n} \mathrm{Inf}_{GL_1^n}^{B} (1_{GL_1} \otimes \cdots \otimes 1_{GL_1})$

$$= \mathrm{Ind}_B^{GL_n}(1_B)$$
$$= \mathbb{C}[GL_n] \otimes_{\mathbb{C}[B]} \mathbb{C} = \mathbb{C}[GL_n/B].$$

(2) V appears as a subrep of $\mathbb{C}[GL_n/B]$ iff $\langle \chi_V, (1_{GL_1})^n \rangle \neq 0$.

But by adjointness of Ind and Res, we have

$$\langle \chi_V, \mathrm{Ind}_B^{GL_n}(1_B) \rangle = \langle \mathrm{Res}_B^{GL_n} \chi_V, 1_B \rangle, \text{ and}$$

the right is nonzero iff 1_B is a subrep of $\mathrm{Res}_B^{GL_n} V$.

QED.

Note that GL_n/B is the variety of complete flags in $\mathbb{F}_2^n$, and the rep $\mathbb{C}[GL_n/B]$ corresponds to the canonical action of GL_n on such. In particular, when $n=2$ $GL_2/B = \mathbb{P}^1(\mathbb{F}_2)$, the projective line.

Prop(4.50): We can choose the PSH isomorphism $\Lambda \xrightarrow{\approx} A(1_{GL_1})$ so that

$$h_n \mapsto 1_{GL_n}.$$

<u>Pf:</u> By thm(3.18), $\chi_{\mathbb{C}[GL_2/B]} = (1_{GL_1})^2$ is a sum of two irreducible characters. One is 1_{GL_2}: $\sum_{c \in GL_2/B} c$ is GL_2-invariant.

Denote the other by St_2, so $(1_{GL_1})^2 = 1_{GL_2} + St_2$.

To prove our prop, by thm's (3.18) and (3.20) it will be sufficient to show that $St_2^{\perp} 1_{GL_n} = 0$ for all n. But

$$\Delta(1_{GL_n}) = \sum_{i+j=n} res_{i,j}(1_{GL_n}) = \sum_{i+j=n} 1_{GL_i} \otimes 1_{GL_j}. \quad \text{So}$$

$$St_2^{\perp} 1_{GL_n} = \sum_{i+j=n} \langle St_2, 1_{GL_i} \rangle 1_{GL_j} = \langle St_2, 1_{GL_2} \rangle 1_{GL_{n-2}} = 0$$

since St_2, 1_{GL_2} are distinct irred reps of GL_2. QED.

Note that the above iso $\Lambda \simeq A(1_{GL_1})$ induces bijections

$$\left\{ \begin{matrix} \text{unipotent characters} \\ \text{of } GL_n \end{matrix} \right\} \longleftrightarrow Par(n), \text{ via Schur functions.}$$

We denote by χ^{λ} the unipotent character corresponding to $\lambda \in Par(n)$.

In particular, $1_{GL_n} = \chi^{(n)}$, as $h_n = s_{(n)}$ in Λ.

(Generalizing the notation St_2, we denote $St_n = \chi^{(1^n)}$, the image of $s_{(1^n)} = e_n$ in $A(1_{GL_1})$. The St_n are called the Steinberg characters)

<u>Example:</u> Consider $GL_2(\mathbb{F}_q)$. By the above proof, GL_2 has exactly two unipotent characters, 1_{GL_2} and St_2, and $1_{GL_2} + St_2 = \chi_{\mathbb{C}[GL_2/B]}$.

As noted earlier, $GL_2/B = \mathbb{P}^1(\mathbb{F}_q)$, so

$$\chi_{\mathbb{C}[\mathbb{P}^1]}(g) = \#(\text{Points of } \mathbb{P}^1(\mathbb{F}_q) \text{ fixed by } g),$$

$$St_2(g) = \#(\text{Points of } \mathbb{P}^1(\mathbb{F}_q) \text{ fixed by } g) - 1$$

In particular, $\dim St_2 = St_2(I_2) = |\mathbb{P}^1(\mathbb{F}_q)| - 1 = q$.

Now consider $GL_2(\mathbb{F}_2)$. One checks that the action of GL_2 on $\mathbb{P}^1(\mathbb{F}_2) = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ sets up an isomorphism $GL_2(\mathbb{F}_2) \simeq \mathfrak{S}_3$.

The conjugacy classes are then

1-cycle	2-cycles	3-cycles
$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}$

Hence, besides 1_{GL_2}, St_2, there is one more irreducible character: it is given by the sign character Sgn of $\mathfrak{S}_3$. By our earlier calculation, $|\mathcal{E}_2|=1$, so Sgn must be cuspidal (the only rep which is both unipotent and cuspidal is 1_{GL_1}, as a fixed point of B is such for any K_{ij}).

We include all this information in the following table:

	1-cycles	2-cycles	3-cycles	
$1_{GL_2} = \chi^{(2)}$	1	1	1	unipotent
$St_2 = \chi^{(1,1)}$	2	0	-1	unipotent
Sgn	1	-1	1	cuspidal