Last time we sketched a proof of

Corollary (4.30) For each of the three towers of groups G_*, $A(G_*)$ is a PSH.

This will allow us to describe $A(G_*)$ in each case.

First, take $G_n = \mathfrak{S}_n$, the symmetric groups

Notation: • $\mathbb{1}_{\mathfrak{S}_n}$, $\mathrm{sgn}_{\mathfrak{S}_n}$ the trivial/sign char's on $\mathfrak{S}_n$.

• Given ~~a~~ $\lambda \in \mathrm{Par}_n$,

$\mathbb{1}_{\mathfrak{S}_\lambda}$, $\mathrm{sgn}_{\mathfrak{S}_\lambda}$ the trivial/sign chars on

$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$, the Young subgroup corr to λ.

• also write $\mathbb{1}_\lambda$ for the indicator fn in $\mathfrak{S}_n$ for cycles of type λ. (Recall the convolution of class fns here: $f, g : H \to \mathbb{C}$, $(f * g)(h) = \cdots$

Thm (i) ~~The irreducible complex characters~~ There is a PSH-isomorphism, the Frobenius characteristic map

$$\underline{ch} : A = A(\mathfrak{S}) \longrightarrow \Lambda \quad \text{s.t.}$$

$$\begin{pmatrix} \mathbb{1}_{\mathfrak{S}_n} \mapsto h_n \\ \mathrm{sgn}_{\mathfrak{S}_n} \mapsto e_n \end{pmatrix}$$

$$\chi^\lambda_\lambda \mapsto s_\lambda$$

$$\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathbb{1}_{\mathfrak{S}_\lambda} \mapsto h_\lambda$$

$$\mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} \mathrm{sgn}_{\mathfrak{S}_\lambda} \mapsto e_\lambda$$

$$\mathbb{1}_\lambda \mapsto \frac{p_\lambda}{z_\lambda}$$

$\lambda = (1^{m_1}, 2^{m_2}, \ldots)$

where $z_\lambda = m_1! 1^{m_1} m_2! 2^{m_2} \cdots$

and the $\{\chi_\lambda\}$, ~~defined~~ are the irreducible characters of $\mathfrak{S}_n$.

(In particular, these are indexed by $\mathrm{Par}(n)$)

(ii) ~~Moreover~~ (The involution on $A(\mathfrak{S}_n)$ given by convolving w/ $\mathrm{sgn}_{\mathfrak{S}_n}$ preserves the $\mathbb{Z}$-sublattice $R(\mathfrak{S}_n)$ of genuine characters.) The direct sum of these involutions, acting on $A = \bigoplus R(\mathfrak{S}_n)$ corresponds to ω under $\underline{ch}$.

part of PSH-auth.

Pf/ Recall 1st that m, Δ as defined are ~~are given by~~ adjoint ~~maps~~ w.r.t. the forms $(\cdot,\cdot)_A$, $(\cdot,\cdot)_{A\otimes A}$, and preserve the genuine characters. Thus for each of the 3 towers, $\Sigma = \coprod_n \mathrm{Irr}(G_n)$ is a PSH-basis for A.

- Taking $\chi \in \Sigma \cap A_n$, one has $\Delta(\chi) = \bigoplus_{i+j=n} \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_i \times \mathfrak{S}_j} \chi$ while $1 \otimes \chi = \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_0 \times \mathfrak{S}_n} \mathbb{1}_{\mathfrak{S}_0} \times \chi = \chi$.
  Thus we have $n=1$ and χ is the trivial char. and $\mathcal{C} = \Sigma \cap p = \mathbb{1}_{\mathfrak{S}_1}$.
- Since Σ has a single primitive elt, there are two isomorphisms $A \to \Lambda$ (3.20). In the notation of (3.18), we have ! sequences $\{e_i'\}, \{h_i'\} \subseteq \Sigma$ w/ $e_2, h_2 \in A_2$. We take $\mathbb{1}_{\mathfrak{S}_2} \mapsto h_2$, $sgn_{\mathfrak{S}_2} \mapsto e_2$, fixing $\underline{ch}$.
- We submit that under $\underline{ch}$, $\mathbb{1}_{\mathfrak{S}_n} \mapsto h_n$, $sgn_{\mathfrak{S}_n} \mapsto e_n$. By (3.18), guaranteeing the unique $\{e_i'\}, \{h_i'\} \subseteq \Sigma$, it is enough to check $sgn_{\mathfrak{S}_2}^{\perp} \mathbb{1}_{\mathfrak{S}_n} = 0 = \mathbb{1}_{\mathfrak{S}_2}^{\perp} sgn_{\mathfrak{S}_n}$.
  - Note that $\mathbb{1}_{\mathfrak{S}_n}$, $sgn_{\mathfrak{S}_n}$ restrict to triv, sign char's on Young subgroups. Thus
  $$\Delta(\mathbb{1}_{\mathfrak{S}_n}) = \sum_{i+j=n} \mathbb{1}_{\mathfrak{S}_i \times \mathfrak{S}_j} = \sum_{i+j=n} \mathbb{1}_{\mathfrak{S}_i} \otimes \mathbb{1}_{\mathfrak{S}_j}$$
  $$sgn_{\mathfrak{S}_2}^{\perp} \mathbb{1}_{\mathfrak{S}_n} = \sum_{i+j=n} (sgn_{\mathfrak{S}_2}, \mathbb{1}_{\mathfrak{S}_i}) \mathbb{1}_{\mathfrak{S}_j} = 0.$$
  To show $\mathbb{1}_{\mathfrak{S}_2}^{\perp} sgn_{\mathfrak{S}_n} = 0$ is similar.
- We can now see the preimages of h_λ, e_λ by writing out induction products and extending linearly.
  (Recall $e_\lambda := e_{\lambda_1} \cdots e_{\lambda_\ell}$, $\lambda = (\lambda_1, \ldots, \lambda_\ell)$)
  (While $\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \mathbb{1}_{\mathfrak{S}_\lambda} = \mathbb{1}_{\mathfrak{S}_{\lambda_1}} \otimes \cdots \otimes \mathbb{1}_{\mathfrak{S}_{\lambda_\ell}}$).

- To look at $\mathbb{1}_{(n)}$, we must work in $A_{\mathbb{C}} = A \otimes_{\mathbb{Z}} \mathbb{C}$ (actually, $A_{\mathbb{Q}}$ will be enough — the rep'ns of $\mathfrak{S}_n$ are defined over $\mathbb{Q}$ - see ex. 4.36)
It is clear that $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_i \times \mathfrak{S}_j} \mathbb{1}_n = 0$ if $i, j > 0$, so this is primitive. By (3.9), the space of primitives is 1-dim in $(A_{\mathbb{Q}})_n$ spanned by p_n, so $\mathbb{1}_{(n)} \mapsto p_n c$
- ~~To see $A_n \mapsto p_\lambda / z_\lambda$, use induction products:~~
On the one hand, $(h_n, p_n)_\Lambda = (h_n, m_n) = 1$ and yet, $(\mathbb{1}_{\mathfrak{S}_n}, \mathbb{1}_{(n)}) = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} \mathbb{1}_{\mathfrak{S}_n}(g) \mathbb{1}_{(n)}(g) = \frac{1}{n!}(n-1)! = \frac{1}{n}$.
So $\mathbb{1}_{(n)} \mapsto p_n / n$ by linearity. To finish, recall $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$ and use induction products.

For part (b), the product of ~~each~~ $\chi \in \mathrm{Irr}(G)$ w/ a linear character is irreducible. This is ~~bijec~~ self-inverse and so a bijection. (Recall a PSH-aut is a graded Hopf-alg morphism which restricts to a bijection of PSH-bases). There is exactly 1 PSH-aut by (3.20) the structure thm for PSH's.

~~To consider~~ Now set $\mathfrak{S}_n = \mathfrak{S}_n[\Gamma]$, we first need the following:

Proposition (4.17) Fix a ~~$\mathbb{C}G$-module~~ $\mathbb{C}[G \ltimes K]$-module V.

(i) For any $\mathbb{C}G$-module U, one has a $\mathbb{C}[G \ltimes K]$-module structure $\Phi(U) = U \otimes V$ determined by

$$k(u \otimes v) = u \otimes kv, \quad g(u \otimes v) = g(u) \otimes g(v) \quad k \in K, \ g \in G.$$

(ii) For any $\mathbb{C}[G \ltimes K]$-module W, one has a $\mathbb{C}G$-mod struct.

$$\Psi(W) = \mathrm{Hom}_{\mathbb{C}K}(\mathrm{Res}^{G \ltimes K}_K V, \mathrm{Res}^{G \ltimes K}_K W) \text{ where}$$

$$g(\varphi) = g \circ \varphi \circ g^{-1}$$

(iii) The maps Φ, Ψ are adjoint; ~~...~~ $\mathrm{Hom}_{\mathbb{C}G}(U, \Psi(W)) \cong \mathrm{Hom}_{\mathbb{C}[G \ltimes K]}(\Phi(U), W)$ under $\varphi \mapsto (u \otimes v \mapsto \varphi(u)(v))$

(iv) One has a $\mathbb{C}G$-module isomorphism
$$(\Psi \circ \Phi)(U) \cong U \otimes \mathrm{End}_{\mathbb{C}K}(\mathrm{Res}_K^{G \times K} V).$$
In particular, if $\mathrm{Res}_K^{G\times K} V$ is a simple $\mathbb{C}K$-module, then $(\Psi \circ \Phi)(U) \cong U$.

~~Now let $\{\rho_1, \dots, \rho_d\}$ be the reps $= \mathrm{Irr}(\Gamma)$~~

Def'n Let $\rho \in \mathrm{Irr}(\Gamma)$, $\chi^\lambda \in \mathrm{Irr}(\mathfrak{S}_n)$, acting on spaces U, V resp. We define a rep'n $\chi^{\lambda,\rho}$ of $\mathfrak{S}_n(\Gamma)$ acting on $U \otimes V^{\otimes n}$ via
$$\sigma(u \otimes v_1 \otimes \cdots \otimes v_n) = \sigma(u) \otimes v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$
$$\gamma(u \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n) = u \otimes \gamma_1 v_1 \otimes \cdots \otimes \gamma_n v_n$$
$$\sigma \in \mathfrak{S}_n, \quad \gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma^n.$$

Thm (4.42) The irreducible $\mathbb{C}\mathfrak{S}_n[\Gamma]$-modules are
$$\chi^{\underline{\lambda}} = \mathrm{Ind}_{\mathfrak{S}_{\underline{\lambda}}[\Gamma]}^{\mathfrak{S}_n[\Gamma]} \left(\chi^{\lambda^{(1)}, \rho_1} \otimes \cdots \otimes \chi^{\lambda^{(d)}, \rho_d} \right)$$
~~with $\underline{\lambda}$ a map function $\mathrm{Irr}(\Gamma) \to \mathrm{Par}$ defined by $\rho_i \mapsto \lambda^{(i)}$ s.t. $\sum_{i=1}^d |\lambda^{(i)}| = n$~~

with $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(d)})$ consisting of d partitions s.t. $\sum |\lambda^{(i)}| = n$. We have $\mathfrak{S}_{\underline{\lambda}} = \mathfrak{S}_{|\lambda^{(1)}|} \times \cdots \times \mathfrak{S}_{|\lambda^{(d)}|}$.

Moreover, one has a PSH-isomorphism
$$A(\mathfrak{S}(\Gamma)) \longrightarrow \Lambda^{\otimes d}$$
with $\chi^{\underline{\lambda}} \longmapsto s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(d)}}$

Pf/ As before, $A(\mathfrak{S}(\Gamma))$ has PSH-basis $\Sigma = \coprod_n \mathrm{Irr}(\mathfrak{S}_n(\Gamma))$

By ~~[illegible]~~ (3.12), writing $\mathcal{C} = \Sigma \cap p$, one has the canonical decomposition

$$A(\mathfrak{S}(\Gamma)) \cong \otimes_{\rho \in \mathcal{C}} A(\mathfrak{S}(\Gamma))(\rho).$$

$A(\rho) = \mathbb{Z}\Sigma(\rho)$

$\Sigma(\rho) := \{\sigma \in \Sigma : (\sigma, \rho^n) \neq 0$ for some $n\}$

- As before, if $\rho \in \mathrm{Irr}\,\mathfrak{S}_n(\Gamma)$ is primitive, then $n=1$ and $\rho \in \{\rho_1, \dots, \rho_d\} = \mathrm{Irr}(\Gamma)$.
- Now fix $\rho = \rho_i$. We must show

$$A(\mathfrak{S}) \xrightarrow{\sim} A(\mathfrak{S}(\Gamma))(\rho) \quad \text{via} \quad \chi^\lambda \mapsto \chi^{\lambda,\rho}.$$

To do so, we make use of (4.17).

~~[illegible]~~ Set V to be the ~~$\mathbb{C}\mathfrak{S}$~~ $\mathbb{C}\mathfrak{S}_n(\Gamma)$-module given by $\chi^{(n),\rho}$ (note $\chi^{(n)}$ is the trivial char.)

We take maps as in the Prop'n:

$$R(\mathfrak{S}_n) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} R(\mathfrak{S}_n(\Gamma)) \quad \text{with}$$

$$\chi \overset{\Phi}{\mapsto} \chi \otimes V, \quad \alpha \overset{\Psi}{\mapsto} \mathrm{Hom}_{\mathbb{C}\Gamma^n}(V, \alpha).$$

Take the direct sum over all n to get maps

$$A(\mathfrak{S}) \rightleftarrows A(\mathfrak{S}(\Gamma))$$

- WTS Φ restricts to an isomorphism on $A(\rho)$.

By adjointness in (4.17), one has

$$(\chi, \Psi(\alpha))_{A(\mathfrak{S})} = (\Phi(\chi), \alpha)_{A(\mathfrak{S}(\Gamma))}, \text{ but use self-duality of } A(\mathfrak{S}), A(\mathfrak{S}(\Gamma)).$$

to see that Φ, Ψ are algebra morph's.

To see that these are coalg. morph's, look at restriction to $\mathfrak{S}_i \times \mathfrak{S}_j$

These are PSH-morphisms ~~[illegible]~~ as they take irred. char's to irred. char's. (look at norms).

Since ~~ψ~~ ρ is simple, so is ~~ψ~~ V, so by (4.17),

$$(\Psi \circ \Phi)(X) = \chi. \text{ for all } \chi \in A,$$

implying that Φ is injective.

- Now show χ simple $\Rightarrow \Phi(\chi)$ simple:

$$(\Phi(\chi), \Phi(\chi))_{A(G(\Gamma))} = ((\Psi \circ \Phi)(\chi), \chi)_{A(G)} = (\chi, \chi)_{A(G)} = 1.$$

- Note that $\Phi(\chi) = \chi \otimes V$ has V as a component when restricting to Γ^n, so by Frobenius reciprocity, $\Phi(\chi)$ is a constituent of $\mathrm{Ind}_{\Gamma^n}^{G_n(\Gamma)} \rho^n$. This means $\mathrm{Im}\,\Phi \subseteq A(G(\Gamma))(\rho)$. The reverse inclusion is obvious.
- Thus Φ is an isomorphism as desired.

Cor (4.43) Given $\underline{\lambda}$ as in the theorem

$$\mathrm{Res}^{G_n(\Gamma)}_{G_{n-1}(\Gamma)\times\Gamma}(\chi^{\underline{\lambda}}) = \sum_{i=1}^{d} \sum_{\substack{\lambda^{(i)}_- \subseteq \lambda^{(i)}: \\ |\lambda^{(i)}/\lambda^{(i)}_-|=1}} \chi^{(\lambda^{(1)}, \dots, \lambda^{(i)}_-, \dots, \lambda^{(d)})} \otimes \rho_i$$

Pf/ Apply the isomorphism of the prev. thm to LHS: We want the component of coproduct $s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(d)}}$ that lies $\Lambda_{n-1} \otimes \Lambda_{\rho_i} \subseteq \Lambda^{\otimes d}$.

Working in each tensor factor Λ, Pieri formula gives the $\Lambda_{|\lambda|-1} \otimes \Lambda_1$-component of $\Delta(s_\lambda)$:

$$\sum_{\substack{\lambda_- \subseteq \lambda \\ |\lambda/\lambda_-|=1}} s_{\lambda_-} \otimes p.$$

Apply this in each factor of the tensor product and sum over i, reversing the isomorphism.