In this section, we want to show that the

algebra and coalgebra

$$A = A(G_*) = \bigoplus_{n \geq 0} R(G_n)$$

are bialgebras.

Recall the Proposition 1.12 of bi-algebras.

For maps $A \otimes A \to A \otimes A$. we ~~ne~~ need to check

that $\Delta \circ m = (m \otimes m) \circ (id \otimes T \otimes id) \circ (\Delta \otimes \Delta)$. (4.20).

$$\begin{tikzcd}
 & A \otimes A \arrow[dl, "\Delta \otimes \Delta"'] \arrow[d, "m"] \\
A \otimes A \otimes A \otimes A \arrow[d, dash, "id \otimes T \otimes id"'] & A \arrow[d, "\Delta"] \\
A \otimes A \otimes A \otimes A \arrow[r, "m \otimes m"'] & A \otimes A
\end{tikzcd}$$

Here, m and Δ are defined as follows:

$m := \mathrm{ind}_{i,j}^{i+j} : A_i \otimes A_j \to A_{i+j}$.

$\Delta := \bigoplus_{i+j=n} \mathrm{res}_{i,j}^{i+j} : A_n \to \bigoplus_{i+j=n} A_i \otimes A_j$.

T is the twist map $A \otimes A \to A \otimes A$ that sends

$a_1 \otimes a_2 \to a_2 \otimes a_1$.

Now, we need some notations

Definition 4.21 : (refer to Page 91).

Definition 4.22 : (refer to Page 92).
parabolic subgroup P_α consists of matrix in the form of
$$\begin{pmatrix} R_{\alpha_1} & \times & \times & \times \\ & R_{\alpha_2} & \times & \times \\ & & \ddots & \vdots \\ & & & R_{\alpha_l} \end{pmatrix}.$$
$P_{(i,j)} = P_{i,j}$ for any $i+j=n$.

Definition 4.23 : (refer to Page 93).

In fact, let $K = {}^gH = gHg^{-1}$
Then $\tau: K \to H$ is the map $k \to g^{-1}kg$.

and $U^\tau = U^g = \mathbb{C}[gHg^{-1}] = \mathbb{C}K$ is a $\mathbb{C}K$-module.

Using homogeneity, in order to check the bialgebra condition (4.10) in the homogeneous component $(A \otimes A)_n$. We only need to verify that ~~condition~~ for each pair of representations U_1 U_2 of G_{r_1} G_{r_2} with $r_1+r_2=n$, and for each (c_1, c_2) with $c_1+c_2=n$, we have

$$\begin{aligned} & \mathrm{res}^n_{c_1 c_2} \left(\mathrm{ind}^n_{r_1 r_2} (U_1 \otimes U_2)\right) \\ (4.21). \quad & \cong \bigoplus_A \left(\mathrm{ind}^{c_1}_{a_{11}, a_{21}} \otimes \mathrm{ind}^{c_2}_{a_{12}, a_{22}}\right)\left(\left(\mathrm{res}^{r_1}_{a_{11}, a_{12}} U_1 \otimes \mathrm{res}^{r_2}_{a_{21}, a_{22}} U_2\right)^{\tau_A^{-1}}\right). \end{aligned}$$

where the direct sum is over all matrices $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $N^{2\times 2}$ with row sums: $a_{11}+a_{12}=r_1$, $a_{21}+a_{22}=r_2$.

and column sums: $a_{11}+a_{21}=c_1$, $a_{12}+a_{22}=c_2$.

τ_A is the twist map:

(4.22). $G_{a_{11},\underline{a_{12},a_{21}},a_{22}} \longrightarrow G_{a_{11},\underline{a_{21},a_{12}},a_{22}}$.

The LHS is a subgroup of G_{r_1,r_2} and the RHS is a subgroup of G_{c_1,c_2}. It's obviously an isomorphism.

We show how to get (4.21) from (4.20).

Given U_1 and U_2. and $r_1+r_2=n$.

the LHS of (4.20) is.

$$\Delta\circ m(U_1\otimes U_2) = \bigoplus_{c_1+c_2=n} \mathrm{res}^n_{c_1 c_2}\left(\mathrm{ind}^n_{r_1 r_2}(U_1\otimes U_2)\right).$$

the RHS of (4.20) is

$(m\otimes m)\circ(id\otimes T\otimes id)\circ(\Delta\otimes\Delta)(U_1\otimes U_2)$

$$\cancel{=\bigoplus_{c_1+c_2=n}\left(\mathrm{ind}^{c_1}_{a_{11},a_{21}}\right.} = \bigoplus_{\substack{a_{11}+a_{12}=r_1\\ a_{21}+a_{22}=r_2}} \left(\mathrm{ind}^{c_1}_{a_{11},a_{21}}\otimes \mathrm{ind}^{c_2}_{a_{12},a_{22}}\right)\left(\left(\mathrm{res}^{r_1}_{a_{11},a_{12}}U_1\otimes \mathrm{res}^{r_2}_{a_{21},a_{22}}U_2\right)\right.$$

$$=\bigoplus_{c_1+c_2=n}\bigoplus_{A}\left(\mathrm{ind}^{c_1}_{a_{11},a_{21}}\otimes \mathrm{ind}^{c_2}_{a_{12},a_{22}}\right)\left(\left(\mathrm{res}^{r_1}_{a_{11},a_{12}}U_1\otimes \mathrm{res}^{r_2}_{a_{21},a_{22}}U_2\right)^{\tau_A^{-1}}\right).$$

Hence we only need to verify (4.21) for any $c_1+c_2=n$.

The reason why we give (4.2) is that it follows by Mackey formula.

Mackey formula: Consider subgroups $H, K < G$ and any $\mathbb{C}H$-module U. ~~If~~ $\{g_1 \cdots g_t\}$ are double coset representatives for $K \backslash G / H$. then

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H U \cong \bigoplus_{i=1}^{t} \mathrm{Ind}^K_{g_i H \cap K} \left(\left(\mathrm{Res}^H_{H \cap K^{g_i}} U \right)^{g_i} \right).$$

For example:

In ~~for~~ the case that $G_n = \mathfrak{S}_n$, the symmetric group.

Let $G = G_n$, $H = G_{(r_1 r_2)}$, $K = G_{(c_1, c_2)}$. $U = U_1 \otimes U_2$

with double coset representatives

$$\{g_1 \cdots g_t\} = \{ w_{A^t} : A \in \mathbb{N}^{2\times 2}. \ A \text{ has row sums } (r_1, r_2) \text{ and column sums } (c_1 \ c_2) \}.$$

*. we will prove this later.

Then for a given double coset, ~~KgH~~ we have

$$KgH = (G_{c_1 c_2}) w_{A^t} (G_{r_1 r_2}).$$

and. $H \cap K^{w_{A^t}} = (G_{r_1, r_2}) \cap (G_{c_1 c_2})^{w_{A^t}}$

which is exactly $G_{a_{11}, a_{12}, a_{21}, a_{22}}$.

and ${}^{w_{A^t}}H \cap K = {}^{w_{A^t}}(G_{r_1, r_2}) \cap (G_{c_1 c_2})$

which is exactly. $G_{a_{11}, a_{21}, a_{12}, a_{22}}$.

Moreover, $\tau_A(g) = w_{A^t} \, g \, w_{A^t}^{-1}$.

So, now we consider the double cosets of $G_\alpha \backslash G_n / G_\beta$.

where α and β are two almost composition of $[n]$.

Definition 4.24. (refer to Page 92).

Example: $n=9$. $\alpha=(4,5)$. $\beta=(3,4,2)$. $A=\begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

$I_1=\{1,2,3,4\}$. $I_2=\{5,6,7,8,9\}$.

$I_{11}=\{1,2\}$. $I_{1,2}=\{3,4\}$ $I_{1,3}=\emptyset$; $I_{21}=\{5\}$. $I_{22}=\{6,7\}$. $I_{23}=\{8,9\}$

$J_1=\{1,2,3\}$. $J_2=\{4,5,6,7\}$. $J_3=\{8,9\}$.

$J_{11}=\{1,2\}$. $J_{12}=\{3\}$, $J_{21}=\{4,5\}$ $J_{2,2}=\{6,7\}$. $J_{3,1}=\emptyset$ $J_{32}=\{8,9\}$

Then w_A is

$$\begin{array}{c} \left(\begin{array}{ccc|cccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{1} & \underline{\underline{2}} & \underline{\underline{5}} & \underline{3} & \underline{4} & \underline{\underline{6}} & \underline{\underline{7}} & \underline{\underline{8}} & \underline{\underline{9}} \end{array}\right) \begin{array}{l} -\ I_1 \\ =\ I_2. \end{array} \\ \qquad J_1 \qquad\qquad\qquad J_2 \qquad\qquad\qquad J_3 \end{array}$$

Remark: Given almost compositions α and β of n. having lengths l and m, and a permutation $w \in \mathfrak{S}_n$. There is a matrix $A \in \mathbb{N}^{l\times m}$ satisfying $w_A = w$ iff. the restriction of w to each J_j and the restriction of w^{-1} to each I_i are increasing. In this case, A is determined by $a_{ij}=|w(J_j)\cap I_i|$.

We can embed $\mathfrak{S}_n$ to $\mathfrak{S}_n[\Gamma]$ for every Γ and into $GL_n(\mathbb{F}_q)$ for every q clearly.

The embedding commute with the group embedding $G_n < G_{n+1}$.

Now, we give the double coset representatives.

Proposition 4.27. The permutations $\{w_A\}$ as A runs over all matrices in $N^{l \times m}$ having row, ~~also~~ column sums α, β give a system of double coset representatives for

① $\mathfrak{S}_\alpha \backslash \mathfrak{S}_n / \mathfrak{S}_\beta$

② $\mathfrak{S}_\alpha[\Gamma] \backslash \mathfrak{S}_n[\Gamma] / \mathfrak{S}_\beta[\Gamma]$

③ $P_\alpha \backslash GL_n / P_\beta$.

Proof: Since $\mathfrak{S}_\alpha[\Gamma] = \mathfrak{S}_\alpha \Gamma^n = \Gamma^n \mathfrak{S}_\alpha$.

the double coset representatives for ①. ~~d~~ also give double coset representatives for ②.

Now, we show that every double coset $\mathfrak{S}_\alpha w \mathfrak{S}_\beta$ contains some w_A. Here are the algorithm

$$w = \left(\begin{array}{ccc|cccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{4} & \underline{\underline{8}} & \underline{2} & \underline{\underline{5}} & \underline{3} & \underline{\underline{9}} & \underline{1} & \underline{\underline{7}} & \underline{\underline{6}} \end{array} \right)$$

$$\Rightarrow w' = \left(\begin{array}{ccc|cccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{2} & \underline{4} & \underline{\underline{8}} & \underline{1} & \underline{3} & \underline{\underline{5}} & \underline{\underline{9}} & \underline{\underline{6}} & \underline{\underline{7}} \end{array} \right) \in w\mathfrak{S}_\beta.$$

That is, replace w by $w' \in w \mathfrak{S}_\beta$ such that the restriction of w' to each J_j ~~are~~ is increasing.

Then

$$w' \Rightarrow w_A = \left(\begin{array}{ccc|cccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 5 & 3 & 4 & 6 & 7 & 8 & 9 \end{array}\right) \in \mathfrak{S}_\alpha w' \subset \mathfrak{S}_\alpha w \mathfrak{S}_\beta$$

That is, replace w' by $w_A \in \mathfrak{S}_\alpha w'$ such that the restriction of w_A^{-1} to each I_i is increasing.

Also, since the values within each set I_i are consecutive, the replacement does not ruin the property that each set $w'(J_j)$ is in increasing order.

Hence, we have w_A is in $\mathfrak{S}_\alpha w \mathfrak{S}_\beta$.

Since $a_{ij}(w_A) = |w_A(J_j) \cap I_i|$.

If $\mathfrak{S}_\alpha w_A \mathfrak{S}_\beta = \mathfrak{S}_\alpha w_B \mathfrak{S}_\beta$, then $\cancel{w_A(J_j) = w_B(J_j)}$.

$w_A(J_j) \cap I_i = w_B(J_j) \cap I_i$, and hence $a_{ij}(w_A) = a_{ij}(w_B)$.

That is, $A = B$. Therefore, we get a system of double coset representative for

$\mathfrak{S}_\alpha \backslash \mathfrak{S}_n / \mathfrak{S}_\beta$ and $\mathfrak{S}_\alpha[\Gamma] \backslash \mathfrak{S}_n[\Gamma] / \mathfrak{S}_\beta[\Gamma]$.

Now, consider $P_\alpha \backslash Gl_n / P_\beta$.

Similarly, we can show that $P_\alpha w_A P_\beta = P_\alpha w_B P_\beta$ implies $A = B$.

the rank $r_{ij}(g)$ of the sub-matrix obtained by restricting g to rows $I_i \sqcup \cdots \sqcup I_l$ and columns $J_1 \cup \cdots \cup J_j$ is constant on double cosets $P_\alpha g P_\beta$ and

$$a_{ij}(w) = r_{ij}(w) - r_{i,j-1}(w) - r_{i+1,j}(w) + r_{i+1,j-1}(w).$$

$$= \left(r_{ij}(w) - r_{i,j-1}(w)\right) - \left(r_{i+1,j}(w) - r_{i+1,j-1}(w)\right).$$

Here $r_{ij}(w) - r_{i,j-1}(w) = \left| w(J_j) \cap (I_i \cup I_{i+1} \cup \cdots \sqcup I_l) \right|$.

Since $|w(J_j) \cap I_i| = |w(J_j) \cap (I_i \cup I_{i+1} \sqcup \cdots \sqcup I_l)|$
$- |w(J_j) \cap (I_{i+1} \cup \cdots \cup I_l)|$.

we have $a_{ij}(w) = |w(J_j) \cap I_i| = (r_{ij}(w) - r_{i,j-1}(w)) - (r_{i+1,j}(w) - r_{i+1,j-1}(w))$.

Thus, we only need to show that every double coset $P_\alpha g P_\beta$ contains som w_A. Since $G_\alpha < P_\alpha$, and we already know that every double coset $G_\alpha w G_\beta$ contains some w_A. It suffices to show that every double coset $P_\alpha g P_\beta$ contains some permutation w. This is already true for ~~some~~ the smaller double cosets BgB where $B = P_{1^n}$ is the Borel subgroup of upper triangular invertible matrices.

That is, we will have the usual Bruhat decomposition

$$GL_n = \bigcup_{w \in \mathfrak{S}_n} BwB.$$

This can be done by column operations gB.
and row operations Bg.
We get g' in the follow way by column operation

①. For the leftmost column, find the bottommost nonzero entry. suppose it's value is a_1. Then multiply g by $\begin{pmatrix} \frac{1}{a_1} & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in B$. to get g_1 That is, scale the column to make this entry a 1.

②. Use that entry in column 1, called pivot to clear out all entries in its row to its right.
That is multiply g_1 by $\begin{pmatrix} 1 & -a_2 & -a_3 & \cdots & -a_n \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \in B$.

Repeat ① and ② for other columns one by one from left to the right.

For example. if $w = \left(\begin{array}{ccc|cccc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{4} & \underline{\underline{8}} & \underline{2} & \underline{\underline{5}} & \underline{3} & \underline{\underline{9}} & \underline{1} & \underline{\underline{7}} & \underline{\underline{6}} \end{array}\right)$.

$g \in BwB$

we can get $g' = \begin{pmatrix} x & x & x & x & x & x & 1 & 0 & 0 \\ x & x & 1 & 0 & 0 & 0 & 0 & \vdots & \vdots \\ x & x & 0 & x & 1 & 0 & \vdots & \vdots & \vdots \\ 1 & 0 & \vdots & 0 & 0 & 0 & & \vdots & \vdots \\ 0 & x & \vdots & 1 & \vdots & 0 & \vdots & 0 & 0 \\ 0 & x & \vdots & 0 & \vdots & x & & x & 1 \\ 0 & x & \vdots & \vdots & \vdots & x & \vdots & 1 & 0 \\ 0 & 1 & \vdots & \vdots & \vdots & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \in gB$. ~~from g~~

Then, we do the row operation. Use the pivot to eliminate all the entries above it in the same column.

For example, for g'.

mutiply $\begin{pmatrix} 1 & & & -g'_{11} & & & \\ & 1 & & -g'_{21} & & & \\ & & 1 & -g'_{31} & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \in B.$ by g'.

then. we can ~~go~~ eliminate the entries above the pivot in column 1. Repeat it for each column.

Finally, we get the permutation matrix for w. $\square$.

Actually. the Bruhat decomposition $Gl_n = \bigcup_{w \in S_n} BwB$ is related to the LPU factorization. It's a fairly general phenmenon.

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