Basic definition

Def: For a group G, a representation is a homomorphism $\varphi: G \to GL(V)$ for some vector space V over a field. We suppose V is finite-dimensional over $\mathbb{C}$ here.

Notice that: a representation of G is the same as a f.d (left) $\mathbb{C}G$-module V.

Here, if S is a set, then $\mathbb{C}S = \mathbb{C}[S]$ denotes the free $\mathbb{C}$-module with basis S.

$\mathbb{C}G$ is the group algebra of G over $\mathbb{C}$.

Def: A $\mathbb{C}G$-module V is completely determined up to isomorphism by its character

$$\chi_V: G \longrightarrow \mathbb{C}$$
$$g \longmapsto \chi_V(g) \triangleq \operatorname{trace}(g: V \to V)$$

The character χ_V is a class function, meaning it is constant on G-conjugacy classes.

The space $R_{\mathbb{C}}(G)$ of class functions $G \to \mathbb{C}$ has a Hermitian, positive definite form

all χ_V

$$(f_1, f_2)_G \triangleq \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Schur's lem: two simple $\mathbb{C}G$-modules V_1, V_2. $\mathrm{Hom}_{\mathbb{C}G}(V_1, V_2) \cong \begin{cases} \mathbb{C} & V_1 \cong V_2 \\ 0 & V_1 \not\cong V_2 \end{cases}$

For any two $\mathbb{C}G$-modules V_1, V_2, $(\chi_{V_1}, \chi_{V_2})_G = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V_1, V_2)$

The set of all irreducible characters $\mathrm{Irr}(G) = \{\chi_V : V \text{ is a simple } \mathbb{C}G\text{-module}\}$ forms an orthonormal basis of $R_{\mathbb{C}}(G)$ with respect to this form, and spans a $\mathbb{Z}$-sublattice

a free $\mathbb{Z}$-module with basis $\mathrm{Irr}(G)$

$R(G) \triangleq \mathbb{Z} \cdot \mathrm{Irr}(G) \leq R_{\mathbb{C}}(G)$ sometimes called the virtual characters of G.

For every $\mathbb{C}G$-module V, the character χ_V belongs to $R(G)$

Def: Define a $\mathbb{C}$-bilinear form $\langle\, ,\, \rangle_G$ on $R_{\mathbb{C}}(G)$ by $\langle f_1, f_2 \rangle_G \triangleq \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g^{-1})$.

$(\, ,\,)_G \neq \langle\, ,\, \rangle_G$. $\langle \chi_{V_1}, \chi_{V_2} \rangle_G = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V_1, V_2)$

$\langle\, ,\, \rangle$ is identical with $(\, ,\,)_G$ on $R(G) \times R(G)$. So we use $\langle\, ,\, \rangle_G$ instead of $(\, ,\,)_G$.

4.1.4. Induction and restriction

Def: Given a subgroup $H<G$ and $\mathbb{C}H$-module U, one can use the fact that $\mathbb{C}G$ is a $(\mathbb{C}G, \mathbb{C}H)$-bimodule to form the induced $\mathbb{C}G$-module.

$$\mathrm{Ind}_H^G U \triangleq \mathbb{C}G \otimes_{\mathbb{C}H} U \qquad \mathbb{C}G \times \mathbb{C}G\otimes_{\mathbb{C}H} U \to \mathbb{C}G\otimes_{\mathbb{C}H} U.$$

The fact that $\mathbb{C}G$ is free as a (right-) $\mathbb{C}H$-module on basis element $\{t_g\}_{gH\in G/H}$.

$$\chi_{\mathrm{Ind}_H^G U}(g) = \frac{1}{H}\sum_{\substack{k\in G:\\ kgk^{-1}\in H}} \chi_U(kgk^{-1})$$

a $\mathbb{C}G$-module V is isomorphic to $\mathrm{Ind}_H^G U$ for some $\mathbb{C}H$-module U iff $\exists$ an H-stable subspace $U\subset V$ having the property that $V=\bigoplus_{gH\in G/H} gU$

The above construction of a $\mathbb{C}G$-module $\mathrm{Ind}_H^G U$ corresponding to any $\mathbb{C}H$-module U is part of a functor Ind_H^G from the category of $\mathbb{C}H$-modules to the category of $\mathbb{C}G$-modules. This functor is called induction.

Def: The restriction operation $\mathrm{Res}_H^G : V \mapsto \mathrm{Res}_H^G V$ restricts a $\mathbb{C}G$-module V to a $\mathbb{C}H$-module.

Frobenius reciprocity asserts the adjointness between Ind_H^G and Res_H^G

$$\mathrm{Hom}_{\mathbb{C}G}(\mathrm{Ind}_H^G U, V) \cong \mathrm{Hom}_{\mathbb{C}H}(U, \mathrm{Res}_H^G V)$$

as a special case $(S=A=\mathbb{C}G, R=\mathbb{C}H, B=U, C=V)$ of the general adjoint associativity $\mathrm{Hom}_S(A\otimes_R B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_S(A,C))$ for S, R two rings, A is an (S,R)-bimodule, B is a left R-module, C is a left S-module.

Def: When H is a subgroup of G, the restriction $\mathrm{Res}_H^G f$ of an $f\in R_{\mathbb{C}}(G)$ is defined as the result of restricting the map $f: G\to\mathbb{C}$ to H. Then $\mathrm{Res}_H^G f\in R_{\mathbb{C}}(H)$.

So Res_H^G is a $\mathbb{C}$-linear map $R_{\mathbb{C}}(G)\to R_{\mathbb{C}}(H)$.

This map restricts to a $\mathbb{Z}$-linear map $R(G)\to R(H)$, since we have $\mathrm{Res}_H^G \chi_V = \chi_{\mathrm{Res}_H^G V}$ for any $\mathbb{C}G$-module V.

4.1.6. Inflation and fixed points.

$\mathrm{Res}^G_H : V \to \mathrm{Res}^G_H V$ restrict a $\mathbb{C}G$-module V to $\mathbb{C}H$-mod

Suppose one has a normal subgroup $K \triangleleft G$. Given a $\mathbb{C}[G/K]$-module U, say defined by the homomorphism $\varphi : G/K \to GL(U)$. The inflation of U to a $\mathbb{C}G$-module $\mathrm{Infl}^G_{G/K} U$ is defined by the composite homomorphism $G \to G/K \xrightarrow{\varphi} GL(U)$. It has the same underlying space U. $\mathrm{Infl}^G_{G/K} U$ is actually a pull back $U \to \mathbb{C}G$-module.

We will later use the fact that when $H < G$ is any other subgroup, one has

(4.10) $\mathrm{Res}^G_H \, \mathrm{Infl}^G_{G/K} U = \mathrm{Infl}^H_{H/H\cap K} \, \mathrm{Res}^{G/K}_{H/H\cap K} U$

(We regard $H/H\cap K$ as a subgroup of G/K, since the canonical homomorphism $H/H\wedge K \to G/K$ is injective)

Def: $V^K \triangleq \{ v \in V : k(v) = v \text{ for } k \in K \}$ (K-fixed space). Inflation turns out to be adjoint to the K-fixed space constrution sending a $\mathbb{C}G$-module V to the $\mathbb{C}[G/K]$-module V^K.

Note that V^K is indeed a G-stable subspace:

Pf: $\forall v \in V^K, g \in G$, $kg(v) = (g \cdot g^{-1}) \cdot k \cdot g(v) = g \cdot (\underbrace{g^{-1} k g}_{\in K \text{ since } K \triangleleft G}(v)) = g(v) \in V^K$ ∎

One has this adjointness (4.11) $\mathrm{Hom}_{\mathbb{C}G}(\mathrm{Infl}^G_{G/K} U, V) = \mathrm{Hom}_{\mathbb{C}[G/K]}(U, V^K)$

Pf: $\forall$ $\mathbb{C}G$-module hom φ on the left, $k\varphi(u) = \varphi(k(u)) = \varphi(u)$ $\forall k \in K$, so that $\varphi \in \mathrm{Hom}_{\mathbb{C}[G/K]}(U, V$

We will also need the following formula for the character χ_{V^K} in terms of the character χ_V: (4.12) $\chi_{V^K}(gK) = \frac{1}{|K|} \sum_{k \in K} \chi_V(gk)$ trace $gk : V \to V$

To see this, note that when one has a $\mathbb{C}$-linear endomorphism φ on a space V that preserve some $\mathbb{C}$-subspace $W \subset V$, if $V \xrightarrow{\pi} W$ is any idempotent projection onto W, then the trace of the restriction $\varphi|_W$ is equal to the trace of $\varphi \circ \pi$ on V.

Applying this to $W = V^K$ and $\varphi = g$, with $\pi = \frac{1}{|K|} \sum_{k \in K} k : V \to V^K$ we can check π is idempotent proje

Another way to restate (4.12) is $\chi_{V^K}(gK) = \frac{1}{|K|} \sum_{h \in gK} \chi_V(h)$ (4.13). equivalent

We have discuss the inflation on modules.

(Inflation and K-fixed space construction can be also defined on class functions.)

set of class functions $G/K \to \mathbb{C}$

For inflation; Inflation $\mathrm{Infl}_{G/K}^{G} f$ of an $f \in R_{\mathbb{C}}(G/K)$ is defined as the composition

back $\quad$ since surjective $\quad$ set of class funct…

$G \twoheadrightarrow G/K \xrightarrow{f} \mathbb{C}$. This is a class function of G and thus lies in $R_{\mathbb{C}}(G)$ $G \to$

(Thus, inflation $\mathrm{Infl}_{G/K}^{G}$ is a $\mathbb{C}$-linear map $R_{\mathbb{C}}(G/K) \to R_{\mathbb{C}}(G)$)

We can check that for every $\mathbb{C}(G/K)$-module U satisfies $\mathrm{Infl}_{G/K}^{G} \chi_U = \chi_{\mathrm{Infl}_{G/K}^{G} U}$,

then $\mathrm{Infl}_{G/K}^{G}$ restricts to a $\mathbb{Z}$-linear map $R(G/K) \to R(G)$. $\quad$ $\chi_U : G/K \to \mathbb{C}$, $gK \mapsto \mathrm{trace}(gK : U \to U)$

We can also use (4.12) or (4.13) as inspiration for defining a "K-fixed space construction" on class functions.

For every class function $f \in R_{\mathbb{C}}(G)$, we define a class function $f^K \in R_{\mathbb{C}}(G/K)$ by

$$f^K(gK) = \frac{1}{|K|}\sum_{k \in K} f(gk) = \frac{1}{|K|}\sum_{h \in gK} f(h),$$ the map $(\cdot)^K : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G/K)$, $f \mapsto f^K$ is $\mathbb{C}$-linear

and restricts to a $\mathbb{Z}$-linear map $R(G) \to R(G/K)$

Then we have $\chi_{V^K} \overset{\text{by (4.12)}}{=} (\chi_V)^K$ for every $\mathbb{C}G$-module V. (relation of K-fixed space between module and class function)

If we take this in (4.11), we obtain $(\mathrm{Infl}_{G/K}^{G} \chi_U, \chi_V)_G = (\chi_U, \chi_V^K)_{G/K}$ for any $\mathbb{C}[G/K]$-module U and any $\mathbb{C}G$-module V (since $\chi_{\mathrm{Infl}_{G/K}^{G} U} = \mathrm{Infl}_{G/K}^{G} \chi_U$, $\chi_{V^K} = (\chi_V)^K$).

By $\mathbb{Z}$-linearity, we have $(\mathrm{Infl}_{G/K}^{G} \alpha, \beta) = (\alpha, \beta^K)_{G/K}$

for any class functions $\alpha \in R_{\mathbb{C}}(G/K)$ and $\beta \in R_{\mathbb{C}}(G)$.

lem 4.8. Let G_1 and G_2 be two groups, and $K_1 < G_1$, and $K_2 < G_2$ be two respective subgroups. Let U_i be a $\mathbb{C}G_i$-module for each $i \in \{1, 2\}$, Then,

$$(4.15) \qquad (U_1 \otimes U_2)^{K_1 \times K_2} = U_1^{K_1} \otimes U_2^{K_2} \quad \text{(as subspaces of } U_1 \otimes U_2\text{)}.$$

Pf: The subgroup $K_1 = K_1 \times 1$ of $G_1 \times G_2$ acts on $U_1 \otimes U_2$.

its fixed points are $(U_1 \otimes U_2)^{K_1} = U_1^{K_1} \otimes U_2$

Similarly. for $K_2 = 1 \times K_2$ of $G_1 \times G_2$ acts on $U_1 \otimes U_2$,

we have $(U_1 \otimes U_2)^{K_2} = U_1 \otimes U_2^{K_2}$

Then we have $(U_1 \otimes U_2)^{K_1 \times K_2} = (U_1 \otimes U_2)^{K_1} \cap (U_1 \otimes U_2)^{K_2}$

$$= (U_1^{K_1} \otimes U_2) \cap (U_1 \otimes U_2^{K_2}) = U_1^{K_1} \otimes U_2^{K_2}$$

last equation is true from the fact that

if P, Q are subspaces of two vector spaces U and V respectively, then

$$(P \otimes V) \cap (U \otimes Q) = P \otimes Q$$

$\blacksquare$

4.2. Three towers of groups.

Here we consider three towers of groups $G_* = (G_0 < G_1 < G_2 < G_3 < \cdots)$

where either

- $G_n = \mathfrak{S}_n$, the symmetric group
- $G_n = \mathfrak{S}_n[\Gamma]$, the wreath product of the symmetric group with some arbitrary finite group Γ.
- $G_n = GL_n(\mathbb{F}_q)$, the finite general linear group

Here, $\mathfrak{S}_n[\Gamma]$ is the semidirect product $\mathfrak{S}_n \ltimes \Gamma^n$ in which $\mathfrak{S}_n$ acts on Γ^n via

$\sigma(\gamma_1, \cdots, \gamma_n) = (\gamma_{\sigma^{-1}(1)}, \gamma_{\sigma^{-1}(2)}, \cdots, \gamma_{\sigma^{-1}(n)})$

For each of the three towers G_*, there are embedding $G_i \times G_j \hookrightarrow G_{i+j}$ and we introduce maps $\mathrm{ind}_{i,j}^{i+j}$ taking $\mathbb{C}[G_i \times G_j]$-modules to $\mathbb{C}G_{i+j}$-modules, as well as maps $\mathrm{res}_{i,j}^{i+j}$ carrying modules in the reverse direction which are adjoint:

(4.18). $\quad \mathrm{Hom}_{\mathbb{C}G_{i+j}}(\mathrm{ind}_{i,j}^{i+j} U, V) = \mathrm{Hom}_{\mathbb{C}[G_i \times G_j]}(U, \mathrm{res}_{i,j}^{i+j} V)$

Def 4.18. For $G_n = \mathfrak{S}_n$, one embeds $\mathfrak{S}_i \times \mathfrak{S}_j$ into $\mathfrak{S}_{i+j}$ as the permutations as the permutations that permute $\{1, 2, \cdots, i\}$, $\{i+1, i+2, \cdots, i+j\}$ separately.

Here one defines $\mathrm{ind}_{i,j}^{i+j} \triangleq \mathrm{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}}$, $\mathrm{res}_{i,j}^{i+j} \triangleq \mathrm{Res}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}}$

For $G_n = \mathfrak{S}_n[\Gamma]$, one embeds $\mathfrak{S}_i[\Gamma] \times \mathfrak{S}_j[\Gamma]$ into $\mathfrak{S}_{i+j}[\Gamma]$ as block monomial matrices whose two diagonal blocks have sizes i, j respectively and define

$\mathrm{ind}_{i,j}^{i+j} \triangleq \mathrm{Ind}_{\mathfrak{S}_i[\Gamma] \times \mathfrak{S}_j[\Gamma]}^{\mathfrak{S}_{i+j}[\Gamma]}$, $\quad \mathrm{res}_{i,j}^{i+j} \triangleq \mathrm{Res}_{\mathfrak{S}_i[\Gamma] \times \mathfrak{S}_j[\Gamma]}^{\mathfrak{S}_{i+j}[\Gamma]}$

For $G_n = GL_n(\mathbb{F}_q)$, denote just GL_n, one embeds $GL_i \times GL_j$ into GL_{i+j} as block diagonal matrices whose two diagonal block have sizes i, j respectively.

Notice that, we can also introduces as an intermediate the parabolic subgroup $P_{i,j}$ consisting of the block upper-triangular matrices of the form $\begin{pmatrix} g_i & L \\ 0 & g_j \end{pmatrix}$ where g_i, g_j lie in GL_i, GL_j, respectively and L in $\mathbb{F}_q^{i \times j}$ is arbitrary.

We have a quotient map $P_{i,j} \to GL_i \times GL_j$ whose kernel $K_{i,j}$ is the set of matrices of the form $\begin{pmatrix} I_i & L \\ 0 & I_j \end{pmatrix}$ with L again arbitrary. One defines

$\mathrm{ind}_{i,j}^{i+j} \triangleq \mathrm{Ind}_{P_{i,j}}^{GL_{i+j}} \mathrm{Infl}_{GL_i \times GL_j}^{P_{i,j}}$, $\quad \mathrm{res}_{i,j}^{i+j} \triangleq (\mathrm{Res}_{P_{i,j}}^{GL_{i+j}}(-))^{K_{i,j}}$

In the case $G_n = GL_n$, the operation $\mathrm{ind}_{i,j}^{i+j}$ is sometimes called parabolic induction or Harish-Chandra induction. The operation $\mathrm{res}_{i,j}^{i+j}$ is essentially the $K_{i,j}$-fixed point construction $V \mapsto V^{K_{i,j}}$. Via (4.7) (4.11), $\mathrm{res}_{i,j}^{i+j}$ is adjoint to $\mathrm{ind}_{i,j}^{i+j}$

Def 4.19. For each of the three towers G_*, define a graded $\mathbb{Z}$-module.

$$A \triangleq A(G_*) = \bigoplus_{n\geq 0} R(G_n)$$

with a bilinear form $(\cdot,\cdot)_A$ whose restriction to $A_n \triangleq R(G_n)$ is the usual form $(\cdot,\cdot)_{G_n}$, and s.t $\Sigma \triangleq \bigcup_{n\geq 0} \mathrm{Irr}(G_n)$ gives an orthonormal $\mathbb{Z}$-basis.

Notice that $A_0 = \mathbb{Z}$ has its basis element 1 equal to the unique irreducible character of the trivial group G_0.

Note that $A_i \otimes A_j = R(G_i) \otimes R(G_j) \cong R(G_i \times G_j)$, then we have candidates for product and coproduct defined by $m \triangleq \mathrm{ind}_{i,j}^{i+j} : A_i \otimes A_j \to A_{i+j}$

$$\Delta \triangleq \bigoplus_{i+j=n} \mathrm{res}_{i,j}^{i+j} : A_n \longrightarrow \bigoplus_{i+j=n} A_i \otimes A_j$$

We first show that m and Δ are adjoint with respect to the forms $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_{A\otimes A}$.

Suppose U, V, W are modules over $\mathbb{C}G_i$, $\mathbb{C}G_j$, $\mathbb{C}G_{i+j}$, respectively, then we can write the $\mathbb{C}[G_i \times G_j]$-module $\mathrm{res}_{i,j}^{i+j} W$ as a direct sum $\bigoplus_k X_k \otimes Y_k$ with X_k being $\mathbb{C}G_i$-modules and Y_k being $\mathbb{C}G_j$-modules, then we have

(4.19) $$\mathrm{res}_{i,j}^{i+j} \chi_W = \sum_k \chi_{X_k} \otimes \chi_{Y_k}.$$

and
$$(m(\chi_U \otimes \chi_V), \chi_W)_A \overset{\mathrm{def}}{=} (\mathrm{ind}_{i,j}^{i+j}(\chi_{U\otimes V}), \chi_W)_A = (\mathrm{ind}_{i,j}^{i+j}(\chi_{U\otimes V}), \chi_W)_{G_{i+j}}$$
$$\overset{(4.18)\,(4.1)}{=} (\chi_{U\otimes V}, \mathrm{res}_{i,j}^{i+j} \chi_W)_{G_i\times G_j} \overset{(4.19)}{=} (\chi_{U\otimes V}, \sum_k \chi_{X_k} \otimes \chi_{Y_k})_{G_i\times G_j}$$
$$= \sum_k (\chi_{U\otimes V}, \chi_{X_k \otimes Y_k})_{G_i\times G_j} \overset{(4.2)}{=} \sum_k (\chi_U, \chi_{X_k})_{G_i} (\chi_V, \chi_{Y_k})_{G_j}$$

$$(\chi_U \otimes \chi_V, \Delta(\chi_W))_{A\otimes A} = (\chi_U \otimes \chi_V, \mathrm{res}_{i,j}^{i+j} \chi_W)_{A\otimes A} \overset{(4.18)}{=} (\chi_U \otimes \chi_V, \sum_k \chi_{X_k} \otimes \chi_{Y_k})_{A\otimes A}$$
$$= \sum_k (\chi_U, \chi_{X_k})_A (\chi_V, \chi_{Y_k})_A = \sum_k (\chi_U, \chi_{X_k})_{G_i} (\chi_V, \chi_{Y_k})_{G_j}$$

Therefore, m, Δ are adjoint maps.

Since Δ is coassociative, then m is associative.