Lec 35 Monday Nov 28

Given PSH A over $k = \mathbb{Z}$

PSH basis Σ

Assume Σ contains a unique primitive element ρ, which is homog deg 1

Goal: display a Hopf algebra iso

$$A \to \Lambda$$

that sends

$$\Sigma \longrightarrow \{s_\lambda\}_{\lambda \in Par}$$

Aside on Λ

LEM Given $n \in \mathbb{N}$ and $\lambda \in Par$

st $$s_\lambda^\perp h_n \neq 0.$$

Then $\exists$ k $(0 \leq k \leq n)$ st

$$\lambda = (k)$$

[so $s_\lambda = h_k$] Moreover

$$h_k^\perp \, h_n = h_{n-k}$$

pf $\exists \mu \in Par$ st

$$(s_\lambda^\perp h_n, \ s_\mu) \neq 0$$

$$\| $$

$$(h_n, \ s_\lambda s_\mu) \qquad \text{Write } h_n = s_\nu \quad \nu = (n)$$

$$h_n = s_\nu$$

$$\|$$

$$c^\nu_{\lambda\mu}$$

$$\|$$

col strict tableaux T of shp ν/λ and content μ st $\mathrm{cnt}(T|\mathrm{cols} \geq j)$ is a partition $\forall j$

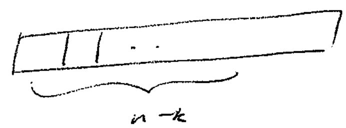
$$\nu = (n): \quad \underbrace{\boxed{\ }\boxed{\ }\cdots\boxed{\ }\boxed{\ }}_{n}$$

Require $\lambda \leq \nu$

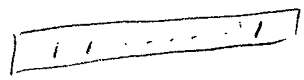
So $\exists\, k \ (0 \leq k \leq n)$ st

$$\lambda = (k)$$

$\nu/\lambda = (n-k)$:



Unique ssyt T is



content μ is $(n-k)$

So $s_\mu = h_{n-k}$

Result follows. $\square$

LEM Given $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}$

st $$\Delta_\lambda^\perp e_n \neq 0.$$

then $\exists$ k $(0 \le k \le n)$ st

$$\lambda = (\underbrace{1, 1, \ldots, 1}_{k})$$

[So $\Delta_\lambda = e_k$] Moreover

$$e_k^\perp e_n = e_{n-k}$$

pf Sim to prev LEM. $\square$

Back to A

We have $A = A(\rho)$, $\Sigma = \Sigma(\rho)$

For $x \in \Sigma$ and $n \in \mathbb{N}$,

$x \in A_n$ iff $(x, \rho^n) \neq 0$

LEM A has finite type

pf For $n \in \mathbb{N}$

$\Sigma \cap A_n$ is a $\mathbb{K}$-basis for A_n

Show $|\Sigma \cap A_n| < \infty$

Write

$$\rho^n = \sum_{x \in \Sigma \cap A_n} \overset{\overset{(x,\rho^n) \neq 0}{\shortparallel}}{\alpha_x} \, x$$

α_x is pos integer

So

$$\infty > (\rho^n, \rho^n) = \sum_{x \in \Sigma \cap A_n} \alpha_x^2$$

$$\geq |\Sigma \cap A_n|$$

$\square$

Since A is graded connected Hopf alg of finite type, the restricted dual A° is a graded Hopf alg iso A.

We now define some elements in A:

$h_n, \quad e_n \qquad n \in \mathbb{N}$

Define

$$h_0 = 1 = e_0 .$$

$$h_1 = \rho = e_1$$

We saw earlier $\exists$ dist $h_2, e_2 \in \Sigma \cap A_2$ st

$$\rho^2 = h_2 + e_2$$

Also

$$\Delta(h_2) = h_2 \otimes 1 + \rho \otimes \rho + 1 \otimes h_2$$

$$\Delta(e_2) = e_2 \otimes 1 + \rho \otimes \rho + 1 \otimes e_2$$

One checks that h_2 is the unique element of $A_2 \cap \Sigma$ st $e_2^{\perp} h_2 = 0$

Moreover $\rho^{\perp} h_2 = h_1$

LEM For $n \geq 2$ $\exists$ unique element h_n in $A_n \cap \Sigma$ st

$$e_2^{\perp} h_n = 0$$

Moreover $$\rho^{\perp} h_n = h_{n-1}$$

pf By ind on n

$n = 2$ ✓

$n \geq 3$: Obs

$$(\rho h_{n-1}, \rho h_{n-1}) = (\rho^{\perp}(\rho h_{n-1}), h_{n-1})$$

[$\rho^{\perp}$ acts as derivation]

$$\overset{\cdot}{=} \Big(\underbrace{\rho^{\perp}(\rho)}_{\substack{\| \\ 1}} h_{n-1} + \rho \underbrace{(\rho^{\perp} h_{n-1})}_{\substack{\| \text{ ind} \\ h_{n-2}}}, \quad h_{n-1} \Big)$$

$$= \underbrace{(h_{n-1}, h_{n-1})}_{\substack{\| \\ 1}} + \underbrace{(\rho h_{n-2}, h_{n-1})}_{\substack{\| \\ \underbrace{(h_{n-2}, \underset{\substack{\| \\ h_{n-2}}}{\rho^{\perp} h_{n-1}})}_{\substack{\| \\ 1}}}}$$

$= 2$

So $\exists$ dist $x, y \in A_n \cap \Sigma$ st

$$p\, h_{n-1} = x + y$$

Find $e_2^\perp x$, $e_2^\perp y$

Obs

$$e_2^\perp x + e_2^\perp y = e_2^\perp (p\, h_{n-1}) \qquad [\, f = e_2 \,]$$

$$= \sum_{(f)} f_1^\perp(p)\, f_2^\perp(h_{n-1})$$

$$= \underbrace{e_2^\perp(p)}_{\substack{\shortparallel \\ 0}} h_{n-1} + \underbrace{p^\perp(p)}_{\substack{\shortparallel \\ 1}} \underbrace{p^\perp(h_{n-1})}_{\substack{\shortparallel \\ h_{n-2}}} + p\, \underbrace{e_2^\perp(h_{n-1})}_{\substack{\shortparallel \\ 0}}$$

$$= h_{n-2} .$$

Switching x, y if nec WLOG

$$e_2^\perp x = 0, \qquad e_2^\perp y = h_{n-2}$$

Define

$$h_n = x$$

So

$$y = p\, h_{n-1} - h_n, \qquad e_2^\perp h_n = 0$$

$$e_2^\perp (p\, h_{n-1}) = h_{n-2}$$

Given $z \in A_n \cap \Sigma$ st

$$e_2^\perp z = 0$$

Show $z = h_n$

Suf to show $(z, \rho h_{n-1}) \neq 0$

Obs $(z, \rho^n) \neq 0$

So

$$0 \neq (z, \rho \rho^{n-1})$$
$$= (\rho^\perp z, \rho^{n-1})$$

So $\rho^\perp z \neq 0$

But $e_2^\perp(\rho^\perp z) = \rho^\perp \underbrace{e_2^\perp z}_{=0} = 0$

So $\rho^\perp z$ is nmo multiple of h_{n-1}

$\downarrow$

Now

$$(z, \rho h_{n-1}) = (\rho^\perp z, h_{n-1})$$
$$\neq 0$$

✓

Check $\rho^\perp h_n = h_{n-1}$

We saw $\rho^\perp h_n = \underbrace{\text{nmo mult}}_{=c}$ of h_{n-1}

show $c = 1$

obs

$c = \left(p^{\perp} h_n, h_{n+1} \right)$

$= \left(\underset{x}{h_n}, \underset{x+1}{p h_{n+1}} \right)$

$= 1$

$\square$

LEM For $n \geq 2$ $\exists$ unique element $e_n \in A_n \cap \Sigma$ st

$$h_2^{\perp} e_n = 0$$

Moreover $p^{\perp} e_n = e_{n-1}$

$\square$

pf Similar to prev LEM

LEM For $n \in \mathbb{N}$ and $z \in \Sigma$

$$z^\perp h_n = 0 \quad \text{unless} \quad z \in \{h_0, h_1, \dots, h_n\}$$

Moreover
$$h_k^\perp h_n = h_{n-k} \qquad (0 \le k \le n)$$

pf Recall $p^\perp h_n = h_{n-1}$

So for $0 \le k \le n$

$$\begin{array}{l} (p^\perp)^k h_n = h_{n-k} \\ \shortparallel \\ (p^k)^\perp h_n \end{array}$$

Write

$$p^k = c h_k + \sum_{\substack{\sigma \in \Sigma \cap A_k \\ \sigma \ne h_k}} c_\sigma \sigma \qquad \begin{array}{l} c, c_\sigma \in \mathbb{Z} \\ c > 0 \\ c_\sigma > 0 \end{array}$$

Obs $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ast$

$$h_{n-k} = (p^k)^\perp h_n = c h_k^\perp h_n + \sum_\sigma c_\sigma \sigma^\perp h_n$$

We have

$$1 = (h_{n-k}, h_{n-k}) = c^2 (h_k^\perp h_n,\ h_k^\perp h_n)$$

$$+\ 2 \sum_\sigma c c_\sigma (h_k^\perp h_n,\ \sigma^\perp h_n)$$

$$+ \sum_{\substack{\sigma, \sigma' \\ \sigma \neq \sigma'}} c_\sigma c_{\sigma'} (\sigma^\perp h_n,\ (\sigma')^\perp h_n)$$

$$+ \sum_\sigma c_\sigma^2 (\sigma^\perp h_n,\ \sigma^\perp h_n)$$

Non Neg ents

$**$

Show $h_k^\perp h_n \neq 0$

We have

$$(p^{n-k})^\perp h_k^\perp h_n = h_k^\perp (p^{n-k})^\perp h_n$$

$$= h_k^\perp h_k$$

$$= 1 \neq 0$$

so $h_k^\perp h_n \neq 0$.

By this and $**$

$$c = 1$$

and $\sigma^\perp h_n = 0 \quad \forall\ \sigma \in \Sigma \cap A_k \setminus h_k$

Now $*$ becomes

$$h_{n-k} = h_k^\perp h_n$$

Result follows.

$\square$

LEM Fix $n \in \mathbb{N}$ and $z \in \Sigma$

$$z^{\perp} e_n = 0 \quad \text{unless} \quad z \in \{e_0, e_1, \ldots, e_n\}$$

Moreover

$$e_k{}^{\perp} e_n = e_{n-k} \qquad (0 \le k \le n)$$

$\square$

pf sim to prev LEM

LEM $\forall n \in \mathbb{N}$

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k} \quad \leftarrow$$

$$\Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}$$

pf $\forall n$ $0 \le k \le n$ and

$\sigma \in A_k \cap \Sigma$, $\tau \in A_{n-k} \cap \Sigma$

$$(\Delta(h_n), \sigma \otimes \tau) = (h_n, \sigma\tau)$$

$$= (\sigma^{\perp} h_n, \tau)$$

0 unless $\sigma = h_k$

$$= (\tau^{\perp} h_n, \sigma)$$

0 unless $\tau = h_{n-k}$.

Also

$$(\Delta(h_n), h_k \otimes h_{n-k}) = (h_n, h_k h_{n-k})$$

$$= (\underbrace{h_k^{\perp} h_n}_{= h_{n-k}}, h_{n-k})$$

$$= 1$$

Result follows. $\square$

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Using the above results we get the desired Hopf alg iso $A \to \Lambda$