Lec 34 Wed Nov 23

Until further notice

A is a PSH over $k = \mathbb{Z}$

PSH basis Σ

$\mathfrak{p}$ = set of prim elements in A

$C = \mathfrak{p} \cap \Sigma$

Next goal: display iso of Hopf algebras

$$A \cong \bigotimes_{\rho \in C} A(\rho)$$

LEM For $\alpha, \beta \in \mathbb{N}^C_{fin}$ st

$$\alpha_\rho \beta_\rho = 0 \qquad \forall \rho \in C$$

"disjoint support"

Then

(i) The following map is a bijection

$$\Sigma(\alpha) \times \Sigma(\beta) \longrightarrow \Sigma(\alpha+\beta) \qquad *$$

$$x \qquad y \longrightarrow xy$$

(ii) The mult map

$$A(\alpha) \otimes A(\beta) \longrightarrow A(\alpha+\beta)$$

$$x \otimes y \longrightarrow xy$$

is a bijection.

pf (i)

For
$$x \in \Sigma(\alpha), \qquad y \in \Sigma(\beta)$$

show
$$xy \in \Sigma(\alpha+\beta)$$

We have
$$xy \in \mathbb{N}\Sigma(\alpha+\beta)$$

show
$$xy \in \Sigma$$

show
$$(xy, xy) = 1$$

obs
$$(xy, xy) = (\Delta(xy), x \otimes y)$$

$$= \sum_{(x)} \sum_{(y)} (x_1 y_1 \otimes x_2 y_2, \; x \otimes y)$$

$$= \sum_{(x)} \sum_{(y)} (x_1 y_1, x)(x_2 y_2, y)$$

Consider a summand

$$(x_1 y_1, x)(x_2 y_2, y)$$

Write

$$x_1 \in \Sigma(\alpha_1), \qquad x_2 \in \Sigma(\alpha_2)$$

$$\alpha_1, \alpha_2 \in \mathbb{N}^C_{fin}, \qquad \alpha_1 + \alpha_2 = \alpha$$

$$y_1 \in \Sigma(\beta_1) \qquad y_2 \in \Sigma(\beta_2)$$

$$\beta_1, \beta_2 \in \mathbb{N}^C_{fin}, \qquad \beta_1 + \beta_2 = \beta$$

So

$$x_1 y_1 \in \mathbb{N}\Sigma(\alpha_1 + \beta_1) \qquad x \in \Sigma(\alpha)$$

$$x_2 y_2 \in \mathbb{N}\Sigma(\alpha_2 + \beta_2) \qquad y \in \Sigma(\beta)$$

$(x_1 y_1, x) = 0$ unless

$$\alpha_1 + \beta_1 = \alpha$$

In which case

$$\beta_1 = 0, \quad \alpha_1 = \alpha, \quad y_1 = 1, \quad x_1 = x$$

Also $(x_2 y_2, y) = 0$ unless

$$\alpha_2 + \beta_2 = \beta$$

In which case

$$\alpha_2 = 0, \quad \beta_2 = \beta, \quad x_2 = 1, \quad y_2 = y$$

So only nonzero summand is

$$\underbrace{(x, x)}_{=1}\underbrace{(y, y)}_{=1} = 1$$

So $(xy, xy) = 1$ ✓

Next show the map $*$ is injective

Given $x, x' \in \Sigma(\alpha)$ $y, y' \in \Sigma(\beta)$

st $xy = x'y'$

show $x = x'$ and $y = y'$

Obs

$$\begin{aligned} 1 &= (xy, xy) \\ &= (xy, x'y') \\ &= \underbrace{(x, x')}_{\substack{= 0 \text{ unless} \\ x = x'}}\underbrace{(y, y')}_{\substack{= 0 \text{ unless} \\ y = y'}} \end{aligned}$$

by sim arg as above

So $x = x'$, $y = y'$ ✓

Show the map $*$ is surjective.

Given $z \in \Sigma(\alpha+\beta)$

Assume $\forall x \in \Sigma(\alpha)$ $\forall y \in \Sigma(\beta)$

$xy \neq z$

So $(z, xy) = 0$

So $(z, \Sigma(\alpha)\Sigma(\beta)) = 0$

So $(z, \mathbb{N}\Sigma(\alpha)\Sigma(\beta)) = 0$

By the def of $\Sigma(\alpha+\beta)$,

$(z, p^{\alpha+\beta}) \neq 0$

$p^{\alpha+\beta} = p^{\alpha} p^{\beta}$

$p^{\alpha} \in \mathbb{N}\Sigma(\alpha)$, $p^{\beta} \in \mathbb{N}\Sigma(\beta)$

So $p^{\alpha+\beta} \in \mathbb{N}\Sigma(\alpha)\Sigma(\beta)$

contr.

So $*$ is surj.

(ii) By (i) $\square$

<u>thm</u> The multiplication map

$$\text{mult} \quad \bigotimes_{\rho \in C} A(\rho) \to A \qquad \star$$

is a Hopf alg iso

pf Show $\star$ is K-module iso.

We have K module iso

$$\bigotimes_{\rho \in C} A(\rho) = \bigotimes_{\rho \in C} \left((A(\rho))_0 + (A(\rho))_1 + \cdots \right)$$

$$\overset{\text{mult}}{\cong} \sum_{\alpha \in \mathbb{N}^C_{fin}} A(\alpha) \qquad (ds)$$

$$= A \quad \checkmark$$

Show $\star$ is Hopf alg morphism.

For notational convenience assume $|C| = 2$

$C = \{\rho, q\}$

Show

$$\overset{R}{\overset{\shortparallel}{A(p)}} \otimes \overset{S}{\overset{\shortparallel}{A(q)}} \longrightarrow A$$

$$m \qquad r \otimes s \longrightarrow rs$$

is Hopf alg morph.

check Δ diag

$$\begin{tikzcd}
R \otimes S \arrow[r, "m"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\
(R\otimes S)\otimes(R\otimes S) \arrow[r, "m \otimes m"'] & A \otimes A
\end{tikzcd}$$

?

Expand

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$$\begin{tikzcd}
R\otimes S \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & A \arrow[ddd, "\Delta"] \\
(R\otimes R)\otimes(S\otimes S) \arrow[d, "1\otimes T\otimes 1"'] & \\
R\otimes S\otimes R\otimes S \arrow[r, "m\otimes m"'] & A\otimes A
\end{tikzcd}$$

?

For $r \in R$ and $s \in S$

$$\begin{tikzcd}
r\otimes s \arrow[r] \arrow[d] & rs \arrow[d] \\
\sum_{(r)}\sum_{(s)} r_1\otimes r_2\otimes s_1\otimes s_2 \arrow[d] & \Delta(rs) = \Delta(r)\Delta(s) \arrow[d, equal] \\
\sum_{(r)}\sum_{(s)} r_1\otimes s_1\otimes r_2\otimes s_2 \arrow[r] & \sum_{(r)}\sum_{(s)} r_1 s_1\otimes r_2 s_2
\end{tikzcd}$$

✓

diagram commutes since Δ is an alg morphism.

Similarly, the diagrams for ε, S commute since ε, S are algebra morphisms

$\square$

Next general goal

Given PSH A over $k = \mathbb{Z}$

PSH basis Σ

Assume Σ contains a unique primitive element ρ

Display a Hopf alg iso

$$A \to \Lambda$$

that sends

$$\Sigma \longrightarrow \{s_\lambda\}_{\lambda \in \mathrm{Par}}$$

Adjusting the grading

Consider the grading

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

$p \in \Sigma$ is homog

Write $p \in A_r$ $\quad 0 \neq r \in \mathbb{N}$

$$A = A(p) = \sum_{n \in \mathbb{N}} (A(p))_n \quad \text{ds}$$

and

$$(A(p))_n \subseteq A_{nr} \quad n \in \mathbb{N}$$

So for $n \in \mathbb{N}$

$$A_n = 0 \quad \text{unless} \quad r \mid n$$

For $n \in \mathbb{N}$ define

$$\tilde{A}_n = A_{rn}$$

then

$$A = \bigoplus_{n \in \mathbb{N}} \tilde{A}_n$$

is a Hopf alg grading

For this grading p has degree 1 ie

$$p \in \tilde{A}_1$$

So WLOG $\quad r = 1$

We wish to find elements in Σ
to be identified with

$$h_n \qquad e_n \qquad n = 1, 2, \dots$$

For notational convenience define

$$h_0 = 1 = e_0$$

Take $\quad h_1 = p = e_1$

Recall $\exists \quad x, y \in \Sigma$ st

$$x + y = p^2$$
$x - y$ is prim

Define $\quad h_2 = x \qquad e_2 = y$

Aside on Λ

LEM For $n \in \mathbb{N}$ and $\lambda \in Par_n$ TFAE

(i) $h_2^\perp s_\lambda = 0$

(ii) $c_{\mu\nu}^\lambda = 0 \quad \forall \mu \in Par_{n-2}$, where $\nu = (2) \in Par_2$

(iii) $\nexists\, \mu \in Par_{n-2}$ st λ/μ is horiz 2-strip

(iv) $\lambda = (1, 1, \ldots, 1)$

Suppose (i)–(iv). then $s_\lambda = e_n$

pf Assume $n \geq 2$ else triv

(i) $\Leftrightarrow$ (ii) $\forall \mu \in Par_{n-2}$

$$
\begin{aligned}
0 &= (h_2^\perp s_\lambda, s_\mu) \\
&= (s_\lambda, h_2 s_\mu) \\
&= (s_\lambda, s_\mu s_\nu) \\
&= c_{\mu\nu}^\lambda
\end{aligned}
$$

$h_2 = s_\nu$

(ii) $\Leftrightarrow$ (iii) $\quad \forall \mu \in Par_{n-2}$

$$c^{\lambda}_{\mu\nu} = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is horiz 2-strip} \\ 0 & \text{else} \end{cases}$$

(iii) $\Leftrightarrow$ (iv) clear. $\square$

LEM For $n \in \mathbb{N}$ and $\lambda \in Par_n$ TFAE

(i) $e_2^{\perp} s_\lambda = 0$

(ii) $c^{\lambda}_{\mu\nu} = 0 \quad \forall \mu \in Par_{n-2}$ where $\nu = (1,1) \in Par_2$

(iii) $\nexists \mu \in Par_{n-2}$ st λ/μ is vertical 2-strip

(iv) $\lambda = (n)$

Suppose (i)–(iv). then $s_\lambda = h_n$.

pf Sim to prev LEM. $\square$