

Lec 34 wed Nov 23

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Until further notice

A is a PSH over $k = \mathbb{Z}$

PSH basis Σ

$P =$ set of prim elements in A

$$C = P \cap \Sigma$$

Next goal: display iso of Hopf algebras

$$A \cong \bigotimes_{p \in C} A(p)$$

LEM For $\alpha, \beta \in \mathbb{N}^C$ for set

$$\alpha_p \beta_p = 0 \quad \forall p \in C$$

"disjoint support"

Then

(i) the following map is a bijection

$$\begin{array}{ccc} \sum |\alpha| & \times & \sum |\beta| & \longrightarrow & \sum |\alpha + \beta| \\ x & & y & \longrightarrow & xy \end{array} \quad *$$

(ii) the mult map

$$\begin{array}{ccc} A(|\alpha|) & \otimes & A(|\beta|) & \longrightarrow & A(|\alpha + \beta|) \\ x & \otimes & y & \longrightarrow & xy \end{array}$$

is a bijection.

pf (i)

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For

$$x \in \Sigma(\alpha), \quad y \in \Sigma(\beta)$$

show

$$xy \in \Sigma(\alpha + \beta)$$

We have

$$xy \in \mathbb{N} \Sigma(\alpha + \beta)$$

show

$$xy \in \Sigma$$

show

$$(xy, xy) = 1$$

obs

$$(xy, xy) = (\Delta(xy), x \otimes y)$$

$$= \sum_{(x)} \sum_{(y)} (x_1 y_1 \otimes x_2 y_2, x \otimes y)$$

$$= \sum_{\alpha_1} \sum_{\gamma_1} (x_1 y_1, x) (x_2 y_2, y)$$

Consider a summand

$$(x_1 y_1, x) (x_2 y_2, y)$$

write

$$x_1 \in \sum(\alpha_1), \quad x_2 \in \sum(\alpha_2)$$

$$\alpha_1, \alpha_2 \in \mathbb{N}_{\text{fin}}^{\mathbb{C}}, \quad \alpha_1 + \alpha_2 = \alpha$$

$$y_1 \in \sum(\beta_1), \quad y_2 \in \sum(\beta_2)$$

$$\beta_1, \beta_2 \in \mathbb{N}_{\text{fin}}^{\mathbb{C}}, \quad \beta_1 + \beta_2 = \beta$$

So

$$x_1 y_1 \in \mathbb{N} \sum(\alpha_1 + \beta_1)$$

$$x_2 y_2 \in \mathbb{N} \sum(\alpha_2 + \beta_2)$$

$$x \in \sum(\alpha)$$

$$y \in \sum(\beta)$$

$$(x_1 y_1, x) = 0 \text{ unless}$$

$$\alpha_1 + \beta_1 = \alpha$$

in which case

$$\beta_1 = 0,$$

$$\alpha_1 = \alpha,$$

$$y_1 = 1,$$

$$x_1 = x$$

$$\text{Also } (x_2 y_2, y) = 0 \text{ unless}$$

$$\alpha_2 + \beta_2 = \beta$$

in which case

$$\alpha_2 = 0, \quad \beta_2 = \beta,$$

$$x_2 = 1, \quad y_2 = y$$

So only non zero summand is

$$\begin{matrix} (x, x) & (y, y) & = & 1 \\ \text{"} & \text{"} & & \\ 1 & 1 & & \end{matrix}$$

$$\text{So } (x, y, x, y) = 1 \quad \checkmark$$

Next show the map \ast is injective

$$\text{Given } x, x' \in \Sigma(\alpha)$$

$$y, y' \in \Sigma(\beta)$$

$$\text{st } xy = x'y'$$

$$\text{show } x=x' \text{ and } y=y'$$

Obs

$$1 = (xy, xy)$$

$$= (xy, x'y')$$

$$= (x, x')(y, y')$$

$$\begin{matrix} \text{"} & \text{"} \\ \text{0 unless} & \text{0 unless} \\ x=x' & y=y' \end{matrix}$$

by sim arg as above

$$\text{So } x=x', y=y' \quad \checkmark$$

Show the map $*$ is surjective.

Given $z \in \Sigma(\alpha + \beta)$

Assume $\forall x \in \Sigma(\alpha) \quad \forall y \in \Sigma(\beta)$

$$xy \neq z$$

$$\text{So } (z, xy) = 0$$

$$\text{So } (z, \Sigma(\alpha) \Sigma(\beta)) = 0$$

$$\text{So } (z, N \Sigma(\alpha) \Sigma(\beta)) = 0$$

By the def of $\Sigma(\alpha + \beta)$,

$$(z, p^{\alpha + \beta}) \neq 0$$

$$p^{\alpha + \beta} = p^\alpha p^\beta$$

$$p^\alpha \in N \Sigma(\alpha), \quad p^\beta \in N \Sigma(\beta)$$

$$\text{So } p^{\alpha + \beta} \in N \Sigma(\alpha) \Sigma(\beta)$$

cont.

So $*$ is surj.

(ii) By (i)

□

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Thm The multiplication map

$$\text{mult} \quad \bigotimes_{p \in C} A(p) \quad \rightarrow A$$

is a Hopf alg iso

pf Show $*$ is K -module iso.

We have K -module iso

$$\bigotimes_{p \in C} A(p) = \bigotimes_{p \in C} \left((A(p))_0 + (A(p))_1 + \dots \right)$$

$$\stackrel{\text{mult}}{\cong} \sum_{\alpha \in \mathbb{N}_{\text{fin}}^C} A(\alpha)$$

(ds)

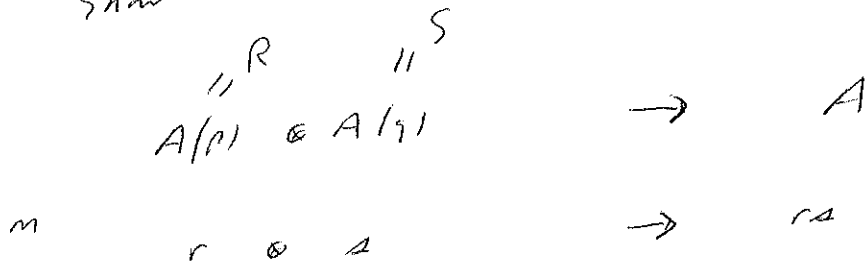
$$= A$$

Show $*$ is Hopf alg morphism.

For notational convenience assume $|C| = 2$

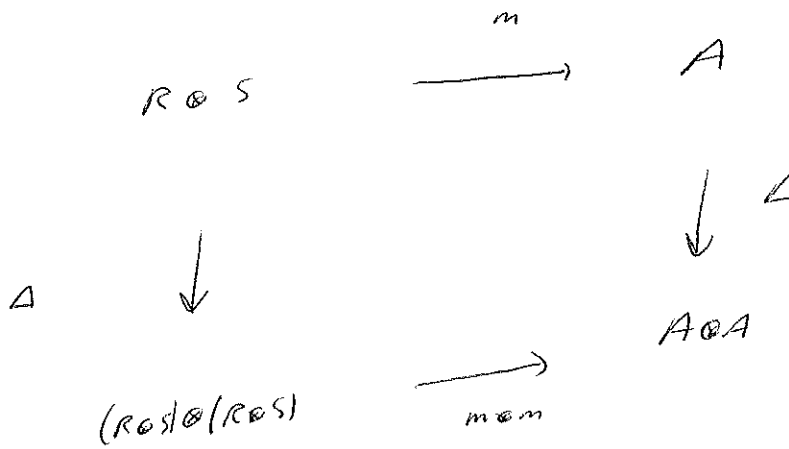
$$C = \{p, q\}$$

show

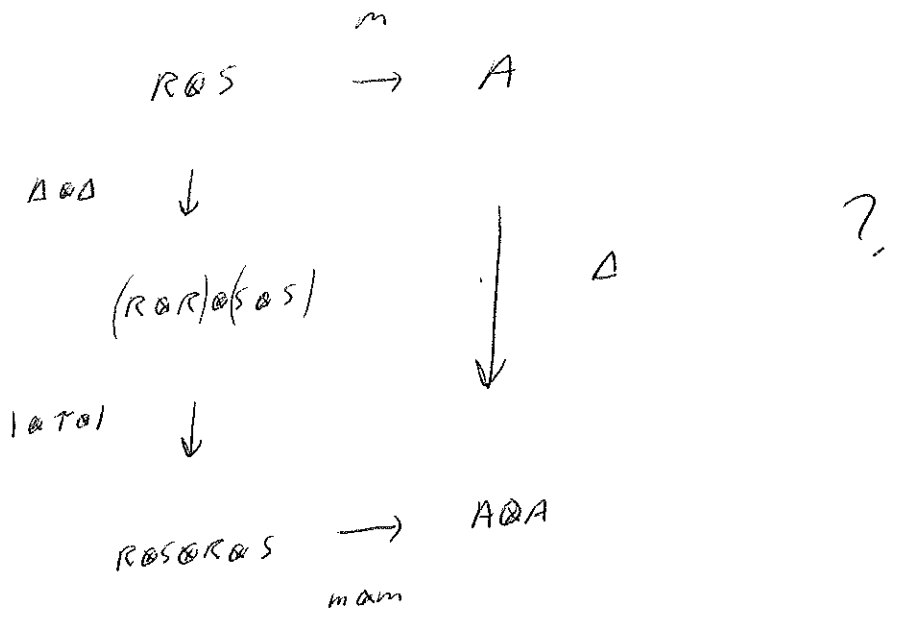


is Hopt alg morph.

check Δ diag



Expand



For $r \in R$ and $s \in S$

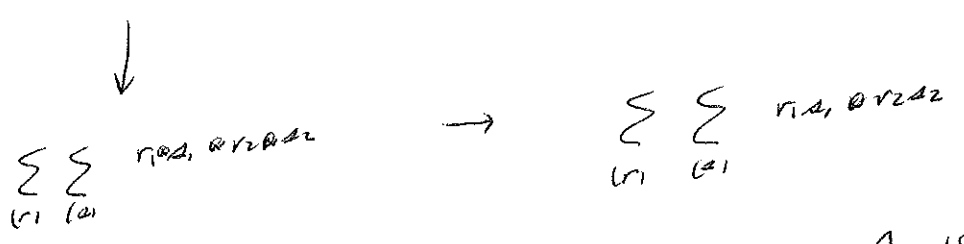
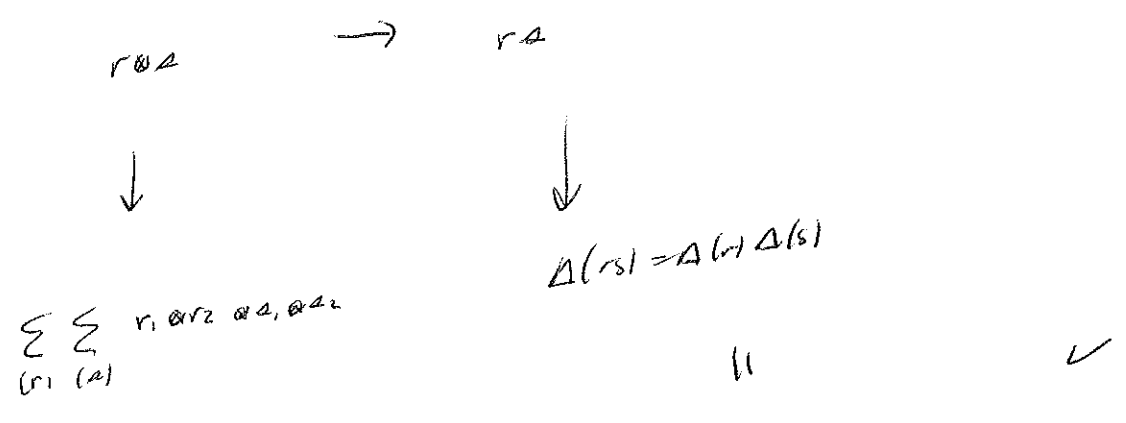


Diagram commutes since Δ is an algebra morphism.

Similarly, the diagrams for ϵ_1, ϵ_2 commute since ϵ_1, ϵ_2 are algebra morphisms.



Next general goal

Given PSH A over $K = \mathbb{Z}$

PSH basis Σ

Assume Σ contains a unique primitive element ρ

Display a Hopf alg iso

$$A \rightarrow \Lambda$$

Haar reads

$$\Sigma \rightarrow \{ \lambda \mid \lambda \in \text{Par} \}$$

Adjusting the grading

Consider the grading

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

$p \in \Sigma$ is homog

Write $p \in A_r$ $0 \neq r \in \mathbb{N}$

$$A = A(p) = \sum_{n \in \mathbb{N}} (A(p))_n \quad ds$$

and

$$(A(p))_n \subseteq A_{nr} \quad n \in \mathbb{N}$$

So for $n \in \mathbb{N}$

$$A_n = 0 \quad \text{unless} \quad r/n$$

For $n \in \mathbb{N}$ define

$$\tilde{A}_n = A_{rn}$$

then

$$A = \bigoplus_{n \in \mathbb{N}} \tilde{A}_n$$

is a Hopf alg grading

For this grading p has degree 1 i.e.

$$p \in \tilde{A}_1$$

So wlog $r=1$

We wish to find elements in Σ
to be identified with

$$h_n \quad e_n \quad n = 1, 2, \dots$$

For notational convenience define

$$h_0 = 1 = e_0$$

Take $h_1 = p = e_1$

Recall $\exists x, y \in \Sigma$ st

$$x + y = p^2$$

$$x - y \text{ is prim}$$

define

$$h_2 = x$$

$$e_2 = y$$

Aside in Λ

LEM For $n \in \mathbb{N}$ and $\lambda \in \text{Par}_n$ TFAE

(i) $h_2^\perp \Delta_\lambda = 0$

(ii) $c_{\mu\nu}^\lambda = 0 \quad \forall \mu \in \text{Par}_{n-2}$, where $\nu = (2) \in \text{Par}_2$

(iii) $\nexists \mu \in \text{Par}_{n-2}$ st λ/μ is horiz 2-strip

(iv) $\lambda = (1, 1, \dots, 1)$

Suppose (i)-(iv). Then $\Delta_\lambda = \mathbb{C}^n$

pf Assume $n \geq 2$ else triv

(i) \Leftrightarrow (ii) $\forall \mu \in \text{Par}_{n-2}$

$$0 = (h_2^\perp \Delta_\lambda, \Delta_\mu)$$

$$= (\Delta_\lambda, h_2 \Delta_\mu)$$

$$= (\Delta_\lambda, \Delta_\mu \Delta_\nu)$$

$$= c_{\mu\nu}^\lambda$$

$$h_2 = \Delta_\nu$$

(iii) \Leftrightarrow (iii) $\forall \mu \in \text{Par}_{n-2}$

$$C_{\mu\nu}^\lambda = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is horiz 2-strip} \\ 0 & \text{else} \end{cases}$$

(iii) \Leftrightarrow (iv) clear. □

LEM For $n \in \mathbb{N}$ and $\lambda \in \text{Par}_n$ TFAE

(i) $e_2^\perp A_\lambda = 0$

(ii) $C_{\mu\nu}^\lambda = 0 \quad \forall \mu \in \text{Par}_{n-2}$ where $\nu = (1,1) \in \text{Par}_2$

(iii) $\nexists \mu \in \text{Par}_{n-2}$ st λ/μ is vertical 2-strip

(iv) $d = (n)$

Suppose (i) - (iv). then $A_\lambda = kn$.

pf Sim to prev LEM. □