Lec 33 Mon Nov 21

Until further notice

A is a PSH over $k = \mathbb{Z}$

PSH basis Σ

$\mathcal{P}$ = set of prim elements in A

$C = \mathcal{P} \cap \Sigma$ Pos $|C| = \infty$

For $\rho \in C$,

$\Sigma(\rho) = \{ x \in \Sigma \,/\, \exists n \in \mathbb{N} \;\; (x, \rho^n) \neq 0 \}$

$A(\rho)$ is k-submodule of A with k-basis $\Sigma(\rho)$

For $a, b \in A$
$a \leq b$ whenever $b - a \in \mathbb{N}\Sigma$

Consider

$$\mathbb{N}^C = \text{set of sequences of natural numbers}$$
$$\{\alpha_p\}_{p\in C}$$

Define

$$\mathbb{N}^C_{fin} = \text{set of sequences } \{\alpha_p\}_{p\in C} \text{ in } \mathbb{N}^C$$
st $\alpha_p \neq 0$ for fin many p

For $\alpha \in \mathbb{N}^C_{fin}$

$$\underbrace{\prod_{p\in C} p^{\alpha_p}}_{\substack{\shortparallel \\ p^\alpha}} \in \mathbb{N}\Sigma$$

Write p^α as $\mathbb{N}$-linear combin of Σ:

$$p^\alpha = \sum_{x\in\Sigma} \beta_x x \qquad \beta_x \in \mathbb{N}$$

Define

$$\Sigma(\alpha) = \{ x\in\Sigma \mid \beta_x \neq 0 \}$$
$$= \{ x\in\Sigma \mid x \leq p^\alpha \}$$

Also define

$$A(\alpha) = \mathbb{Z}\,\Sigma(\alpha)$$

Note that p^α $\alpha \in \mathbb{N}^C_{fin}$
are not always

LEM We have

(i) $$\Sigma = \bigcup_{\alpha \in \mathbb{N}^C_{fin}} \Sigma(\alpha)$$ (disj union)

(ii) $$A = \sum_{\alpha \in \mathbb{N}^C_{fin}} A(\alpha)$$ (orthog dir sum)

pf (i) Fn dist $\alpha, \beta \in \mathbb{N}^C_{fin}$ show

$$\Sigma(\alpha) \cap \Sigma(\beta) = \phi$$

Suppose $x \in \Sigma(\alpha) \cap \Sigma(\beta)$

$(x, \rho^\alpha) \neq 0$ $x \leq \rho^\alpha$

$(x, \rho^\beta) \neq 0$ $x \leq \rho^\beta$

Now $(\rho^\alpha, \rho^\beta) \neq 0$ contr. ✓

Fn $x \in \Sigma$ display $\alpha \in \mathbb{N}^C_{fin}$ st $x \in \Sigma(\alpha)$

Write $x \in A_n$ Use ind on n

$n = 0, 1$ ✓

$n \geq 2$: Recall

$$A_n = I_n^2 + P_n$$ (ods)

First assume

$$(x, I_n^2) = 0$$

then $x \in P_n$

So $x \in P \cap \Sigma = C$

Define $\alpha \in \mathbb{N}^C_{fin}$ st

$\alpha_x = 1, \quad \alpha_y = 0 \quad \forall y \in C \setminus x$

So $x = p^\alpha$

$(x, x) \neq 0$ so $x \in \Sigma(\alpha)$ ✓

Next assume

$$(x, I_n^2) \neq 0$$

Recall $I_n^2 = \sum_{i=1}^{n-1} A_i A_{n-i}$

$\exists\ i \quad 1 \le i \le n-1 \qquad \exists z \in A_{n-i} \cap \Sigma$

$\exists y \in A_i \cap \Sigma,$

st $(x, yz) \neq 0$

By ind

$\exists \beta \in \mathbb{N}^C_{fin}$ st $y \in \Sigma(\beta)$

$\exists \gamma \in \mathbb{N}^C_{fin}$ st $z \in \Sigma(\gamma)$

So

$$y \le p^{\beta}, \qquad z \le p^{\gamma}$$

Now

$$x \le yz \le p^{\beta+\gamma}$$

So $\quad x \in \Sigma(\alpha) \quad$ where $\alpha = \beta + \gamma$

$\square$

(ii) By (i)

LEM Fn $\alpha, \beta \in \mathbb{N}^C_{fin}$

(i) $\Sigma(\alpha) \quad \Sigma(\beta) \subseteq \mathbb{N}\,\Sigma(\alpha+\beta)$

(ii) $A(\alpha) \quad A(\beta) \subseteq A(\alpha+\beta)$

pf (i) Fn

$x \in \Sigma(\alpha) \qquad y \in \Sigma(\beta)$

$x \leq p^{\alpha} \qquad y \leq p^{\beta}$

$xy \in p^{\alpha+\beta}$

Fn $z \in \Sigma$ st $z \leq xy$

$z \leq xy \leq p^{\alpha+\beta}$

so $z \in \Sigma(\alpha+\beta)$

Now

$$xy = \sum_{\substack{z \in \Sigma \\ z \leq xy}} \underbrace{(xy, z)}_{\in \mathbb{N}} z$$

$$\in \mathbb{N}\,\Sigma(\alpha+\beta)$$

(ii) By (i) $\square$

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LEM Fix $\gamma \in \mathbb{N}^C_{fin}$

$$\Delta(A(\gamma)) \subseteq \sum_{\substack{\alpha,\beta \in \mathbb{N}^C_{fin} \\ \alpha+\beta=\gamma}} A(\alpha) \otimes A(\beta) \qquad (*)$$

pf Fix $x \in A(\gamma)$

show $\Delta(x) \in$ RHS of $(*)$

WLOG $x \in \Sigma(\gamma)$

Write $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$

Given summand

$$\underset{\substack{\| \\ y}}{x_1} \otimes \underset{\substack{\| \\ z}}{x_2}$$

WLOG $y, z \in \Sigma$

$\exists\ \alpha \in \mathbb{N}^C_{fin}$ st $y \in \Sigma(\alpha)$

$\exists\ \beta \in \mathbb{N}^C_{fin}$ st $z \in \Sigma(\beta)$

show $\alpha+\beta=\gamma$

We have $0 \neq (\Delta(x), y \otimes z) = (x, yz)$

$yz \in \Sigma(\alpha+\beta)$

$\Sigma(\gamma)$, $\Sigma(\alpha+\beta)$ are orthog unless $\gamma = \alpha+\beta$

So $\gamma = \alpha+\beta$

$\square$

LEM Fix $p \in C$

$A(p)$ is a subalgebra of A

pf Define $\alpha \in \mathbb{N}^C_{fin}$ st

$$\alpha_x = \begin{cases} 1 & \text{if } x = p \\ 0 & \text{if } x \neq p \end{cases} \quad x \in C$$

So $p^\alpha = p$

Fix $n \in \mathbb{N}$ and $x \in \Sigma$

$$(x, p^n) \neq 0 \iff x \in \Sigma(n\alpha)$$

So $$A(p) = \sum_{n \in \mathbb{N}} A(n\alpha)$$

Fix $r, s \in \mathbb{N}$
$$A(r\alpha)\, A(s\alpha) \subseteq A((r+s)\alpha)$$

Also $$A(0\alpha) = A_0 = k\mathbb{1}$$

Result follows. $\square$

LEM For $p \in C$ the algebra
$A(p)$ is graded with homog components
$$(A(p))_n = A(p) \cap A_{rn} = A(n\alpha) \qquad n \in \mathbb{N}$$
where $p \in A_r$
and $\alpha \in \mathbb{N}^C_{fin}$ is from prev LEM.

$\square$

pf clear.

LEM The graded alg $A(p)$ is connected with
$$(A(p))_0 = A_0 = K1$$

$\square$

pf clear.

LEM For $p \in C$

$A(p)$ is a Hopf subalgebra of A

Moreover

$$A(p) = \sum_{n \in \mathbb{N}} A(n\alpha)$$

is a Hopf alg grading

pf For $n \in \mathbb{N}$

$$\Delta(A(n\alpha)) \subseteq \sum_{i=0}^{n} A(i\alpha) \otimes A((n-i)\alpha)$$

Also for $n \geq 1$

$$\varepsilon(A(n\alpha)) \subseteq \varepsilon(A_{nr}) = 0$$

Result follows. $\square$

LEM For $\rho \in C$

$A(\rho)$ is a PSH with PSH basis

$\Sigma(\rho)$

pf We saw $A(\rho)$ is graded connected Hopf alg with k-basis $\Sigma(\rho)$.

This basis is PSH since Σ is PSH. $\square$

LEM For $\rho \in C$

ρ is the unique primitive element in $\Sigma(\rho)$

pf $\rho \in \Sigma(\rho)$ since $(\rho, \rho) = 1 \neq 0$

Given prim $x \in \Sigma(\rho)$

show $x = \rho$.

Suppose $x \neq \rho$

then $(x, \rho) = 0$, $x, \rho \in P$ $\star$

So $\rho^n x^m$ $n, m \in \mathbb{N}$

are not orthog.

$\star$ contains x and all powers of ρ. So

$(x, \rho^n) = 0$ $n \in \mathbb{N}$

Now $x \notin \Sigma(\rho)$, contr. $\square$

So $x = \rho$.

— o —

We have shown that for $\rho \in C$, $A(\rho)$ is a PSH whose PSH basis has a unique prim element.