Lec 32 Friday Nov 18

Next goal

Given PSH $A$ over $k = \mathbb{Z}$

show how $A$ is related to $\Lambda$

Recall For the Hopf alg $\Lambda$ over $k = \mathbb{Z}$

For $\lambda, \mu \in$ Par Hall $( , )$ satisfies

$$(p_\lambda, p_\mu) = \delta_{\lambda,\mu} \, z_\lambda$$

where

$$z_\lambda = m_1! \, 1^{m_1} \, m_2! \, 2^{m_2} \cdots$$

$m_i$ = # parts of $\lambda$ equal to $i$

Obs $z_\lambda = 1$ iff $\lambda = (1)$

So $p_1$ is unique power sym function that is a unit vector rel $( , )$

So $p_1$ is unique primitive element in the PSH-basis

$s_\lambda \quad \lambda \in$ Par

LEM For the Hopf alg $\Lambda$ over $k = \mathbb{Z}$

For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$

$$(s_\lambda, p_1^n) = \text{positive integer}$$

pf Ind on $n$

$n=0$: $s_\phi = 1$
$p_1^0 = 1$
$(1,1) = 1$ ✓

$n=1$ $s_\square = p_1$
$(p_1, p_1) = 1$ ✓

$n \geq 2$ $s_\lambda$ not prim

$$\Delta(s_\lambda) = \sum_{\mu,\nu \in \mathrm{Par}} c^\lambda_{\mu,\nu}\, s_\mu \otimes s_\nu$$

$\exists$ $\tilde{\mu}, \tilde{\nu} \in \mathrm{Par}$ st

$\tilde{\mu} \neq \lambda$, $\tilde{\nu} \neq \lambda$, $c^\lambda_{\tilde{\mu}\tilde{\nu}} \neq 0$

let $r = |\tilde{\mu}|$, $t = |\tilde{\nu}|$

So $1 \leq r, t \leq n-1$, $r + t = n$

Obs

$$(s_\lambda, p_1^n) = (s_\lambda, p_1^r p_1^t)$$

$$= (\Delta(s_\lambda), p_1^r \otimes p_1^t)$$

$$= \sum_{\mu,\nu \in \mathrm{Par}} c^\lambda_{\mu,\nu} \underbrace{(s_\mu \otimes s_\nu, p_1^r \otimes p_1^t)}_{\shortparallel \\ (s_\mu, p_1^r)(s_\nu, p_1^t)}$$

$(s_\mu, p_1^r)$: 0 unless $|\mu| = r$

$(s_\nu, p_1^t)$: 0 unless $|\nu| = t$

$$= \sum_{\mu \in \mathrm{Par}_r} \sum_{\nu \in \mathrm{Par}_t} c^\lambda_{\mu,\nu} (s_\mu, p_1^r)(s_\nu, p_1^t)$$

$c^\lambda_{\mu,\nu}$: NN int

$(s_\mu, p_1^r)$, $(s_\nu, p_1^t)$: pos int

not all summands are 0 since
the summand for $(\mu, \nu) = (\tilde{\mu}, \tilde{\nu})$ is non 0

$= $ pos int

$\square$

Given PSH $A$ over $k = \mathbb{Z}$

Recall PSH basis $\{\sigma_\lambda\}_\lambda$

Call this basis $\Sigma$

Let

$C$ = set of primitive elements in $\Sigma$

For $\rho \in C$ define

$$\Sigma(\rho) = \{\sigma \in \Sigma \mid \exists\, n \in \mathbb{N} \text{ st } (\sigma, \rho^n) \neq 0\}$$

$$A(\rho) = \mathbb{Z}\text{-span of } \Sigma(\rho)$$

Going to show

- $A(\rho)$ is Hopf subalg of $A$ that is iso $\Lambda$
- $\exists$ Hopf alg iso
$$A \simeq \bigotimes_{\rho \in C} A(\rho)$$

## Motivation

Before going through the formal pf, let us consider how to get started

Recall for $\sigma_\lambda, \sigma_\mu, \sigma_\nu \in \Sigma$

$$(\sigma_\lambda, \sigma_\mu \sigma_\nu) = a^\lambda_{\mu\nu} = (\Delta(\sigma_\lambda), \sigma_\mu \otimes \sigma_\nu)$$

is a nonneg integer

Recall grading $\quad A = \bigoplus_{n \in \mathbb{N}} A_n$

$$P_n = A_n \cap P \qquad n \in \mathbb{N}$$

$$P = \sum_{n \in \mathbb{N}} P_n \qquad P_0 = 0, \quad P_1 = A_1$$

For $n \geq 1$ consider $\rho \in C$ with $\rho \in P_n$

We have $\quad (\rho, \rho) = 1$

Consider $\rho^2$

$$\rho^2 \in A_{2n} \cap I^2 = I^2_{2n}$$

$$\Delta(\rho^2) = (\rho \otimes 1 + 1 \otimes \rho)^2$$

$$= \rho^2 \otimes 1 + 2\rho \otimes \rho + 1 \otimes \rho^2$$

$$(\rho^2, \rho^2) = (\Delta(\rho^2), \rho \otimes \rho)$$

$$= 2$$

Write $p^2$ as $K$-linear comb of $\Sigma$

$$p^2 = \sum_\lambda a^\lambda_{\mu\mu} \sigma_\lambda \qquad p = \sigma_\mu$$

$$2 = (p^2, p^2) = \sum_\lambda (a^\lambda_{\mu\mu})^2$$

$a^\lambda_{\mu\mu} = 1$ for 2 dist $\lambda$'s, and 0 otherwise

$p^2 =$ sum of 2 dist elements in $\Sigma$, call them $x, y$

$$p^2 = x + y \qquad x, y \in A_{2n}$$

| $( , )$ | $x$ | $y$ |
|---|---|---|
| $x$ | 1 | 0 |
| $y$ | 0 | 1 |

$\Delta(p^2) =$

| | $\otimes$ |
|---|---|
| $x$ | 1 |
| $y$ | 1 |
| $2p$ | $p$ |
| 1 | $x$ |
| 1 | $y$ |

Given $a, b \in \Sigma$ st $a \neq 1$, $b \neq 1$

Find $(x, ab)$, $(y, ab)$ *

* are nonneg integers

Obs $(x, ab) + (y, ab) = (p^2, ab)$

$= (\Delta(p^2), a \otimes b)$

$= \begin{cases} 2 & \text{if } a = p = b \\ 0 & \text{else} \end{cases}$

$(x, ab) = 0 = (y, ab)$ unless $a = p = b$

Also

$(x, p^2) = (x, x+y) = 1$

$(y, p^2) = (y, x+y) = 1$

So $x, y \in A(p)$

Now

$\Delta(x) = x \otimes 1 + p \otimes p + 1 \otimes x$

$\Delta(y) = y \otimes 1 + p \otimes p + 1 \otimes y$

So $\Delta(x-y) = (x-y) \otimes 1 + 1 \otimes (x-y)$

$x - y \in P_{2n}$

Write

$$p_1 = p$$

$$p_2 = x - y$$

$$(p_2, p_2) = 2$$

$$p_2 \notin \Sigma$$

Also

$$(p_2, p_1^2) = (x-y, x+y) = 0$$

| $(\,,\,)$ | $p_1^2$ | $p_2$ |
|---|---|---|
| $p_1^2$ | 2 | 0 |
| $p_2$ | 0 | 2 |

transition matrices are

| | $p_1^2$ | $p_2$ |
|---|---|---|
| $x$ | 1 | 1 |
| $y$ | 1 | -1 |

For the Hopf alg $\Lambda$

| | $\rho_1^2$ | $\rho_2$ |
|---|---|---|
| $s_{\cdot\cdot}$ | 1 | 1 |
| $s_{:}$ | 1 | $-1$ |

So in our Hopf alg iso $\Lambda \to A(\rho)$
we expect

$s_{\cdot\cdot} \to x$

$s_{:} \to y$

This ends the motivation. We now begin our formal pf.

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Recall the elements of $\Sigma$ are mutually orthog

So the elements of $C$ are mut orthog and primitive

LEM For the PSH $A$ over $k = \mathbb{Z}$

Given mut orthog prim elements

$$p_1, p_2, \ldots, p_n$$

then the elements

$$p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \qquad r_1, r_2, \ldots, r_n \in \mathbb{N}$$

are mut orthog

pf For

$$r_1 \ldots r_n, \quad s_1 \ldots s_n \in \mathbb{N}$$

show

$$\left( p_1^{r_1} \cdots p_n^{r_n}, \, p_1^{s_1} \cdots p_n^{s_n} \right) = 0$$

unless $r_i = s_i \quad 1 \le i \le n$

Use ind on $\ast$

$$r_1 + \cdots + r_n + s_1 + \cdots + s_n$$

$\ast = 0$ ✓

$\ast > 0$ WLOG $\exists\, i \;\; s_i \neq 0$

WLOG $s_1 \neq 0$

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$$\left( p_1^{r_1} \cdots p_n^{r_n},\ p_1^{s_1} \cdots p_n^{s_n} \right) = \left( \Delta(p_1^{r_1} \cdots p_n^{r_n}),\ p_1 \otimes \underbrace{p_1^{s_1 - 1} p_2^{s_2} \cdots p_n^{s_n}}_{= q} \right)$$

$$\left[ \begin{aligned} \Delta(p_1^{r_1} \cdots p_n^{r_n}) &= \prod_{j=1}^{n} \left( \sum_{i_j=0}^{r_j} \binom{r_j}{i_j} p_j^{i_j} \otimes p_j^{r_j - i_j} \right) \\ &= \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \cdots \sum_{i_n=0}^{r_n} \binom{r_1}{i_1} \cdots \binom{r_n}{i_n} p_1^{i_1} \cdots p_n^{i_n} \otimes p_1^{r_1 - i_1} \cdots p_n^{r_n - i_n} \end{aligned} \right]$$

$$= \sum_{i_1=0}^{r_1} \cdots \sum_{i_n=0}^{r_n} \binom{r_1}{i_1} \cdots \binom{r_n}{i_n} \underbrace{\underbrace{\left( p_1^{i_1} \cdots p_n^{i_n},\ p_1 \right)}_{= 0 \text{ unless } i_1 = 1,\ i_2 = 0,\ \vdots} \text{ By ind } \underbrace{\left( p_1^{r_1 - i_1} \cdots p_n^{r_n - i_n},\ q \right)}_{= 0 \text{ unless } r_1 - i_1 = s_1 - 1,\ r_2 - i_2 = s_2,\ \vdots}}_{= 0 \text{ unless } r_1 = s_1,\ r_2 = s_2,\ \vdots}$$

$$= 0 \text{ unless } \begin{cases} r_1 = s_1 \\ r_2 = s_2 \\ \vdots \\ r_n = s_n \end{cases}$$

$\square$

<u>Notation</u> For any abel gp $G$

view $G$ as $\mathbb{Z}$ module

For a subset $S \subseteq G$

$$\mathbb{Z}S = \text{subgp of } G \text{ gen by } S$$

$$= \left\{ \sum_{s \in S} \alpha_s s \;\middle|\; \alpha_s \in \mathbb{Z} \right\}$$

$$\mathbb{N}S = \left\{ \sum_{s \in S} \alpha_s s \;\middle|\; \alpha_s \in \mathbb{N} \right\}$$

"monoid gen by $S$"

$\mathbb{N}S$ is closed under $+$, and contains $0$

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For the PSH $A$ over $k = \mathbb{Z}$

For $a, b \in A$ define

$a \leq b$ whenever $b - a \in \mathbb{N}\Sigma$

$\leq$ is partial order

Observe

- For $\sigma_\mu, \sigma_\nu \in \Sigma$

$$\sigma_\mu \sigma_\nu = \sum_\lambda \underbrace{a^\lambda_{\mu\nu}}_{\in \mathbb{N}} \sigma_\lambda \qquad \in \mathbb{N}\Sigma$$

So for $x, y \in \mathbb{N}\Sigma$

$$xy \in \mathbb{N}\Sigma$$

- For $a, b, c \in A$ st

$$a \geq b \quad \text{and} \quad c \geq 0$$

$$ac \geq bc$$

check:

$$a - b \in \mathbb{N}\Sigma$$
$$c \in \mathbb{N}\Sigma$$

So $ac - bc = (a-b)c \in \mathbb{N}\Sigma$ ✓