Lec 31 Wed Nov 16

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For K arb

Given K-module V

Recall Hopf alg $\mathrm{Sym}(V)$

Δ, ε, S are K-alg morphisms and

$\Delta(x) = x \otimes 1 + 1 \otimes x$

$\varepsilon(x) = 0$ $\quad \forall x \in V$

$S(x) = -x$

So $V \subseteq P$

Do we have $V = P$? Not in general

ex K = field with char $p > 0$

$\dim V = 1$ basis x

$\mathrm{Sym}(V) = K[x]$

$$\Delta(x^p) = (\Delta(x))^p$$
$$= (x \otimes 1 + 1 \otimes x)^p$$
$$= \sum_{i=0}^{p} \binom{p}{i} x^i \otimes x^{p-i}$$

Since p is prime

$$\binom{p}{i} = 0 \qquad 1 \le i \le p-1$$

So

$$\Delta(x^p) = x^p \otimes 1 + 1 \otimes x^p$$

$$x^p \in P$$

$$V \subsetneq P$$

LEM Assume $\mathbb{Q}$ is a subring of K

Given K-module V

Then for the Hopf alg $Sym(V)$

$$P = V$$

pf Write $H = Sym(V)$

H has grading

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

$$H_0 = K1 \qquad H_1 = V$$

For $n \in \mathbb{N}$ define

$$P_n = H_n \cap P$$

Then
$$P = \sum_{n \in \mathbb{N}} P_n$$

and $\quad P_0 = 0, \qquad P_1 = H_1 = V$

For $n \geq 2$ show
$$P_n = 0$$

For $s \in \mathbb{N}$ define k-module hom
$$\pi_s : H \to H_s$$
that acts on H_r as $\begin{cases} id & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \qquad r \in \mathbb{N}$

Consider the composition
$$\varphi : H_n \xrightarrow{\Delta} H_n \otimes H_0 + H_{n-1} \otimes H_1 + \cdots + H_0 \otimes H_n \xrightarrow{\pi_{n-1} \otimes \pi_1} H_{n-1} \otimes H_1 \xrightarrow{mult} H_n$$

<u>claim</u> $\quad \varphi(x) = nx \qquad \forall x \in H_n$

pf of $\quad$ WLOG $\quad x = x_1 x_2 \cdots x_n \qquad x_i \in V \quad 1 \leq i \leq n.$

$$x_1 x_2 \cdots x_n \xrightarrow[\Delta]{} \prod_{i=1}^{n} (x_i \otimes 1 + 1 \otimes x_i)$$

$$\xrightarrow[\pi_{n-1} \otimes \pi_1]{} \sum_{i=1}^{n} x_1 \cdots \hat{x_i} \cdots x_n \otimes x_i \xrightarrow[mult]{} n\, x_1 x_2 \cdots x_n$$

claim proved ✓

Now suppose $P_n \neq 0$

Pick $0 \neq x \in P_n$

Find $\varphi(x)$

$$x \xrightarrow[\Delta]{} x \otimes 1 + 1 \otimes x \xmapsto[\pi_{n-1} \otimes \pi_1]{} 0$$

So $\varphi(x) = 0$

$\quad \shortparallel$

$\quad nx$

n^{-1} exists in k so

$x = 0$ cont

$\square$

Assume $\mathbb{Q}$ is a subring of K

Recall the Hopf alg Λ over K

We saw earlier

$p_1, p_2, p_3, \ldots$

are alg indep and generate Λ

So

$$\Lambda \cong K[p_1, p_2, p_3, \ldots]$$
$$\cong \operatorname{Sym}(V)$$

where
$$V = \sum_{n=1}^{\infty} K p_n$$

COR With above notation

$p_1, p_2, p_3, \ldots$

is a K-basis for the set of primitive elements of Λ. $\square$

Prop Given

- field k char 0
- graded connected Hopf alg A over k
- the sum

$$I = P + I^2$$

is direct

Then the incl map $P \to A$ induces a Hopf alg iso

$$\theta : \mathrm{Sym}(P) \to A$$

Pf Consider grading $A = \bigoplus_{n \in \mathbb{N}} A_n$

Recall $I = \bigoplus_{n=1}^{\infty} A_n$

For $n \in \mathbb{N}$ define

$$P_n = P \cap A_n$$

$P_0 = 0$ since $P \subseteq I$

For $x \in P$ write

$$x = \sum_{n \in \mathbb{N}} x_n \qquad x_n \in A_n$$

One checks $x_n \in P$ for $n \in \mathbb{N}$

Therefore

$$P = \sum_{n \in \mathbb{N}} P_n$$

For $n \in \mathbb{N}$ define

$$I^2_n = I^2 \cap A_n$$

We have

$$I^2_n = A_1 A_{n-1} + A_2 A_{n-2} + \cdots + A_{n-1} A_1$$

Also $I^2_0 = 0, \quad I^2_1 = 0$

$$I^2 = \sum_{n=2}^{\infty} I^2_n$$

We assume the sum $I = P + I^2$ is direct.

So for $n \geq 1$,

$$A_n = P_n + I_n^2 \qquad \text{(dir sum)}$$

Since the alg A is commutative, the incl map $P \to A$ induces an algebra morphism

$$\theta : \operatorname{Sym}(P) \to A$$

<u>claim</u> θ is surjective

<u>pf</u> Let $\langle P \rangle$ = subalg of A gen by P
= image of $\operatorname{Sym}(P)$ under θ

show $\langle P \rangle = A$

For $n \in \mathbb{N}$ show $A_n \subseteq \langle P \rangle$

Use ind on n

<u>Case $n = 0$:</u> $A_0 = k1 \quad 1 \in \langle P \rangle$ ✓

<u>Case $n \geq 1$:</u>

$$A_n = P_n + I_n^2 = \underset{\substack{\cap \\ P \subseteq \langle P \rangle}}{P_n} + \underbrace{\sum_{i=1}^{n-1} A_i A_{n-i}}_{\subseteq \langle P \rangle \text{ by ind}}$$

claim proved

<u>claim 2</u> θ is a Hopf alg morphism

<u>pf cl</u> For the Hopf alg A,

Δ, ε, S are algebra morphisms st

$\Delta(p) = p \otimes 1 + 1 \otimes p$

$\varepsilon(p) = 0$ $\forall p \in P$

$S(p) = -p$

For the Hopf alg $\mathrm{Sym}(P)$,

Δ, ε, S are alg morphisms st

$\Delta(p) = p \otimes 1 + 1 \otimes p$

$\varepsilon(p) = 0$ $\forall p \in P$

$S(p) = -p$

The map θ sends $p \to p$ $\forall p \in P$.

the claim follows.

The Hopf alg $Sym(P)$ is graded as follows:

For $n \in \mathbb{N}$ an element in P_n is called

homogeneous of degree n.

Given homog $p_1, p_2, \dots, p_r$ in P consider the

monomial

$$p_1 p_2 \cdots p_r \in Sym(P)$$

Define

$$\deg(p_1 p_2 \cdots p_r) = \sum_{i=1}^{r} \deg(p_i)$$

For $n \in \mathbb{N}$ define

$$Sym_n(P) = K\text{-span of the monomials with deg } n$$

So $\quad P_n \subseteq Sym_n(P)$

By constr

$$Sym(P) = \sum_{n \in \mathbb{N}} Sym_n(P) \qquad (\text{dsum})$$

This is a Hopf alg grading. Moreover

$$\theta(Sym_n(P)) = A_n \qquad n \in \mathbb{N}$$

We show θ is injective.

Define
$$J = \ker(\theta)$$

show $J = 0$

For $n \in \mathbb{N}$ define
$$J_n = J \cap \mathrm{Sym}_n(P)$$

For $x \in J$ write
$$x = \sum_{n \in \mathbb{N}} x_n \qquad x_n \in \mathrm{Sym}_n(P)$$

Apply θ
$$0 = \sum_{n \in \mathbb{N}} \underbrace{\theta(x_n)}_{\in A_n}$$

For $n \in \mathbb{N}$
$$\theta(x_n) = 0$$
so $x_n \in J_n$

Therefore
$$J = \sum_{n \in \mathbb{N}} J_n$$

show $J_n = 0$ for $n \in \mathbb{N}$

use ind on n

<u>Case $n=0$</u> $\quad J_0 = 0$ since $Sym_0(P) = k1$

and $\theta(1) = 1$

<u>Case $n \geq 1$</u> $\quad$ Abbr

$$S_j = Sym_j(P) \qquad j \in \mathbb{N}$$

Fix $x \in J_n$ show $x = 0$

Since $x \in J_n \subseteq S_n$

$$\underbrace{\Delta(x) - x \otimes 1 - 1 \otimes x}_{= y} \in S_1 \otimes S_{n-1} + S_2 \otimes S_{n-2} + \cdots + S_{n-1} \otimes S_1$$

Find $(\theta \otimes \theta)(y)$

$\theta \otimes \theta$ sends

$$\Delta(x) \longrightarrow \Delta(\underbrace{\theta(x)}_{= 0}) = 0$$

$$x \otimes 1 \longrightarrow \underbrace{\theta(x)}_{= 0} \otimes 1 = 0$$

$$1 \otimes x \longrightarrow 1 \otimes \underbrace{\theta(x)}_{= 0} = 0$$

So $(\theta \otimes \theta)(y) = 0$

By ind

$$0 = J_0 = J_1 = \cdots = J_{n-1}$$

So θ is injective on

$$S_0 + S_1 + \cdots + S_{n-1}$$

So $\theta \otimes \theta$ is injective on

$$S_1 \otimes S_{n-1} + S_2 \otimes S_{n-2} + \cdots + S_{n-1} \otimes S_1 \qquad *$$

But $y \in *$ and $(\theta \otimes \theta)(y) = 0$ so

$$y = 0$$

So

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

x is prim in $Sym(P)$

Since $char(K) = 0$, each prim element of $Sym(P)$ is contained in P

So $x \in P$

By constr θ acts on P as the incl map $P \to A$

So θ is inj on P. We have $\theta(x) = 0$ so

$$x = 0$$

We have shown θ is inj.

The result follows $\square$

COR With ref to above Prop the Hopf alg A is both commutative and cocommutative

pf Since this is true of $\mathrm{Sym}(P)$ $\square$