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Lec 30 Monday Nov 14

III Positive self dual Hopf algebras

Def let A denote a graded connected Hopf alg over $k = \mathbb{Z}$, with distinguished k-basis $\{\sigma_\lambda\}_\lambda$ of homogeneous elements.

Call A a <u>positive self dual Hopf alg</u> (PSH)

whenever

- the same structure constants $a^\lambda_{\mu\nu}$ describe mult

$$\sigma_\mu \sigma_\nu = \sum_\lambda a^\lambda_{\mu\nu} \sigma_\lambda$$

and comult

$$\Delta(\sigma_\lambda) = \sum_{\mu,\nu} a^\lambda_{\mu\nu} \sigma_\mu \otimes \sigma_\nu$$

- $a^\lambda_{\mu\nu} \geq 0 \quad \forall \lambda, \mu, \nu$

Call $\{\sigma_\lambda\}_\lambda$ the <u>PSH-basis</u> for A

Ex $\quad A = \Lambda, \quad \sigma_\lambda = s_\lambda, \quad a^\lambda_{\mu\nu} = c^\lambda_{\mu\nu}$

Assume A is a PSH

Consider grading $A = \bigoplus_{n \in \mathbb{N}} A_n$

Recall $A_0 = k\,1$

$\exists$ unique k-basis element σ_λ contained in A_0

Write $\lambda = \phi$

$\exists$ $0 \neq \alpha \in k$ st

$\sigma_\phi = \alpha 1$

We have $\sigma_\phi^2 = \alpha \sigma_\phi$

So $\alpha = a_{\phi\phi}^{\phi} > 0$

Recall $\Delta(1) = 1 \otimes 1$

So $\Delta(\sigma_\phi) = \frac{1}{\alpha} \sigma_\phi \otimes \sigma_\phi$

So $\frac{1}{\alpha} = a_{\phi\phi}^{\phi} = \alpha$

So $\alpha \in \{1, -1\}$

$\alpha = 1$ since $\alpha > 0$

So $\sigma_\phi = 1$

Recall counit ε satisfies

$$\ker(\varepsilon) = \bigoplus_{n=1}^{\infty} A_n \quad (= I)$$

Recall $\forall x \in I$

$$\Delta(x) - x \otimes 1 - 1 \otimes x \in I \otimes I$$

Let

P = set of primitive elements in A

P is a K-submodule of A

LEM We have $P \subseteq I$

pf For $x \in P$ write

$$x = a1 + y \qquad a \in K \quad y \in I$$

Show $a = 0$

$$\underbrace{\Delta(y)}_{\Delta(x) - a\Delta(1)} - \underbrace{y \otimes 1}_{x \otimes 1 - a 1 \otimes 1} - \underbrace{1 \otimes y}_{1 \otimes x - a 1 \otimes 1} \in I \otimes I$$

$\Delta(x) - a\Delta(1)$

$=$

$$\underbrace{x \otimes 1 + 1 \otimes x - a(1 \otimes 1)}_{}$$

$a \; 1 \otimes 1$

$1 \otimes 1 \notin I \otimes I$ So $a = 0$ $\square$

Assume A is a PSH

Endow A with k-bilin form $(\, , \,)$

that makes $\{ \sigma_\lambda \}_\lambda$ orthonormal

So $(1,1) = 1$ and

$(A_r, A_s) = 0$ if $r \neq s$ $\forall r, s \in \mathbb{N}$

Also, endow $A \otimes A$ with k-bil form $(\, , \,)$

that makes $\{ \sigma_\lambda \otimes \sigma_\mu \}_{\lambda, \mu}$ orthonormal.

Obs

$$(a \otimes b, a' \otimes b') = (a, a')(b, b') \qquad \forall a, a', b, b' \in A$$

LEM With above notation

(i) $(a, bc) = (\Delta(a), b \otimes c)$ $\forall\, a, b, c \in A$

(ii) $(a, 1) = \varepsilon(a)$ $\forall a \in A$

pf (i) $\forall\, \lambda, \mu, \nu$

$$\underbrace{\Big(\sigma_\lambda, \underbrace{\sigma_\mu \sigma_\nu}_{= \sum_{\lambda'} a^{\lambda'}_{\mu\nu} \sigma_{\lambda'}}\Big)}_{= a^\lambda_{\mu\nu}} \stackrel{?}{=} \underbrace{\Big(\underbrace{\Delta(\sigma_\lambda)}_{= \sum_{\mu', \nu'} a^{\lambda}_{\mu'\nu'} \sigma_{\mu'} \otimes \sigma_{\nu'}}, \sigma_\mu \otimes \sigma_\nu\Big)}_{= a^\lambda_{\mu\nu}}$$

ok

(ii) $\forall \lambda$

$$(\sigma_\lambda, 1) \stackrel{?}{=} \varepsilon(\sigma_\lambda)$$

Assume $\lambda = \phi$, else both sides 0

$$(\sigma_\phi, 1) \stackrel{?}{=} \varepsilon(\sigma_\phi)$$

$$(\sigma_\phi, 1) = (1, 1) = 1 \qquad \varepsilon(\sigma_\phi) = \varepsilon(1) = 1$$

ok

$\square$

Assume A is a PSH

For $n \in \mathbb{N}$ the form $(,)$ on A_n induces a k-module hom

$$\varphi_n : A_n \to A_n^* \qquad a \to \varphi_n(a)$$

st $\varphi_n(a)(b) = (a, b)$ $\qquad a, b \in A_n$

$\exists$ k-module hom

$$\varphi : A \to A^o$$

that acts on A_n as φ_n for $n \in \mathbb{N}$

The map φ is injective.

By self duality

φ is k-alg hom.

Via φ. identify A with a k-subalg of A^o

If A has finite type then φ is k-alg iso

and therefore $A = A^o$.

Aside on bilinear forms

Until further notice k is arb

Def Given graded k-modules

$$V = \bigoplus_{n \in \mathbb{N}} V_n$$

$$W = \bigoplus_{n \in \mathbb{N}} W_n$$

Given k-bil form

$$(,) \quad V \times W \to k$$

Call $(,)$ <u>graded</u> whenever

$$(V_r, W_s) = 0 \text{ if } r \neq s \qquad r, s \in \mathbb{N}$$

Given k-modules V, W

Given sym k-bil forms $\quad *$

$$(\ ,\ )\quad V \times V \to k$$

$\quad **$

$$(\ ,\ )\quad W \times W \to k$$

Recall $\exists$ sym k-bil form $\quad ***$

$$(\ ,\ )\quad V\otimes W \times V\otimes W \to k$$

$$v\otimes w \quad v'\otimes w' \to (v,v')(w,w')$$

LEM Assume $*$, $**$ are graded. Then $***$ is graded.

pf Write

$$V = \bigoplus_{n\in\mathbb{N}} V_n \qquad W = \bigoplus_{n\in\mathbb{N}} W_n$$

Fa $n \in \mathbb{N}$

$$(V\otimes W)_n = \sum_{i=0}^{n} V_i \otimes W_{n-i}$$

For $r, s \in \mathbb{N}$ st $r \neq s$ show

$$\Big(\underbrace{(V \otimes W)_r}_{\substack{\| \\ V_0 \otimes W_r + \cdots + V_r \otimes W_0}}, \underbrace{(V \otimes W)_s}_{\substack{\| \\ V_0 \otimes W_s + \cdots + V_s \otimes W_0}} \Big) \overset{?}{=} 0$$

For $0 \le i \le r$ and $0 \le j \le s$ show

$$\left(V_i \otimes W_{r-i}, V_j \otimes W_{s-j} \right) = 0$$

yes since

$i \neq j$ or $r-i \neq s-j$

$\square$

DEF Given any bialg A over K

Given a sym K-bil form

$$(,) : A \times A \to K$$

Call A self-dual wrt $(,)$ whenever

- $(a, bc) = (\Delta(a), b \otimes c) \quad \forall a, b, c \in A$
- $(a, 1) = \varepsilon(a) \quad \forall a \in A$

Ex A PSH is self-dual wrt the bil form that makes the PSH basis orthonormal

Prop Assume $K = \mathbb{Z}$ or $K = \mathbb{Q}$

Given a graded connected Hopf alg A over K

Given a graded pos def K-bil form

$(,) : A \times A \to K$ with respect to which A is self dual.

then $\forall x \in I$ TFAE

(i) x is primitive

(ii) x is orthog to $I^2 = \sum_{a,b \in I} K ab$

pf Write $y = \Delta(x) - x \otimes 1 - 1 \otimes x$

So $y \in I \otimes I$.

$\forall a, b \in I$

$$\begin{aligned}(y, a \otimes b) &= (\Delta(x) - x \otimes 1 - 1 \otimes x, a \otimes b) \\ &= \underbrace{(\Delta(x), a \otimes b)}_{\substack{\| \\ (x, ab)}} - (x, a)\underbrace{(1, b)}_{\substack{\| \\ 0}} - \underbrace{(1, a)}_{\substack{\| \\ 0}}(x, b) \\ &= (x, ab)\end{aligned}$$

Now

$$\begin{aligned} x \text{ is prim} &\iff y = 0 \\ &\iff (y, a \otimes b) = 0 \quad \forall a, b \in I \\ &\iff (x, ab) = 0 \quad \forall a, b \in I \\ &\iff x \text{ orthog } I^2 \end{aligned}$$

$\square$

COR Ref to above Prop

(i) $P \cap I^2 = 0$

(ii) Assume $k = \mathbb{Q}$ then

$$I = P + I^2$$ (orthog dir sum)

pf (i) By the Prop and since $(,)$ is pos def.

(ii) View A as vectorspace over $\mathbb{Q}$. By the Prop

P is the orthog complement of I^2 in I $\square$

LEM Given a graded connected Hopf alg A st $P \cap I^2 = 0$.

Then the algebra A is commutative.

pf Consider the grading

$$A = \bigoplus_{n \in \mathbb{N}} A_n \qquad A_0 = K1$$

For $i, j \in \mathbb{N}$ show

$$[A_i, A_j] = 0$$

Use ind on $i+j$

Case $i+j = 0$ ok since $A_0 = K1$

Case $ij \geq 1$ Assume $i \geq 1$ and $j \geq 1$ else clear

Pick $x \in A_i$, $y \in A_j$

show $xy - yx = 0$

$x, y \in I$ so $xy - yx \in I^2$

show $xy - yx \in P$

show $\Delta(xy - yx) - (xy - yx) \otimes 1 - 1 \otimes (xy - yx) \stackrel{?}{=} 0$ $\ast$

$$\| $$

$\Delta(x)\Delta(y) - \Delta(y)\Delta(x)$

We have

$$\underbrace{\Delta(x) - x\otimes 1 - 1\otimes x}_{\substack{\| \text{ dfy} \\ X}} \in A_1\otimes A_{i-1} + A_2\otimes A_{i-2} + \cdots + A_{i-1}\otimes A_1$$

$$\underbrace{\Delta(y) - y\otimes 1 - 1\otimes y}_{\substack{\| \text{ dfy} \\ Y}} \in A_1\otimes A_{j-1} + \cdots + A_{j-1}\otimes A_1$$

By ind

$$[X, Y] = 0,$$

$$[x\otimes 1, Y] = 0, \qquad [1\otimes x, Y] = 0$$

$$[X, y\otimes 1] = 0, \qquad [X, 1\otimes y] = 0$$

Now eval LHS of $*$ using

$$\Delta(x) = x\otimes 1 + 1\otimes x + X,$$

$$\Delta(y) = y\otimes 1 + 1\otimes y + Y$$

After cancelling get

$$\text{LHS of } * = [X, Y] + [x\otimes 1, Y] + [1\otimes x, Y] + [X, y\otimes 1] + [X, 1\otimes y]$$
$$= 0$$

$\square$