Lec 3 Monday Sept 12 9/12/16

We continue to discuss coalgebras.

Recall: Given coalgebras C, C'

then $C \otimes C'$ is a coalgebra with

$$\Delta : C \otimes C' \xrightarrow[\Delta \otimes \Delta']{} C \otimes C \otimes C' \otimes C' \xrightarrow[1 \otimes \tau \otimes 1]{} C \otimes C' \otimes C \otimes C'$$

$$\varepsilon : C \otimes C' \xrightarrow[\varepsilon \otimes \varepsilon']{} k \otimes k \longrightarrow k$$

$$1 \otimes 1 \longrightarrow 1$$

Def Given coalgebras C, C'.

A map $\varphi : C \to C'$ is a <u>coalgebra morphism</u> whenever φ is a K-module hom that makes these diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "\varphi \otimes \varphi"'] & C' \otimes C' \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\varepsilon"'] & C' \arrow[d, "\varepsilon'"] \\ K \arrow[r, "id"'] & K \end{tikzcd}$$

Given A, an algebra or coalgebra

When is ε

		(if A is alg) alg morphism	(if A is coalg) coalg morphism
(if A is alg)	m		
	u	III	II
(if A is coalg)	Δ	I require $\Delta(ab) = \Delta(a)\Delta(b)$ $\Delta(1_A) = 1_A \otimes 1_A$	IV
	ε	require $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ $\varepsilon(1_A) = 1$	

9/12/16

Investigate II

when is $m : A \otimes A \to A$ a coalg morphism?

Require

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\Delta_{A\otimes A}"'] & A \arrow[d, "\Delta"] \\ A \otimes A \otimes A \otimes A \arrow[r, "m \otimes m"'] & A \otimes A \end{tikzcd}$$

For $a, b \in A$

$$\begin{tikzcd} a \otimes b \arrow[rr] \arrow[d] & & ab \arrow[d] \\ \Delta(a) \otimes \Delta(b) & & \Delta(ab) \end{tikzcd}$$

$$\Delta(a) \otimes \Delta(b) = \sum_{(a)} a_1 \otimes a_2 \otimes \sum_{(b)} b_1 \otimes b_2$$

$$\begin{tikzcd} \sum_{(a)} a_1 \otimes a_2 \otimes \sum_{(b)} b_1 \otimes b_2 \arrow[d, "1 \otimes \tau \otimes 1"] \\ \sum_{(a)} \sum_{(b)} a_1 \otimes b_1 \otimes a_2 \otimes b_2 \arrow[r] & \sum_{(a)} \sum_{(b)} a_1 b_1 \otimes a_2 b_2 \end{tikzcd}$$

$$\Delta(a) \otimes \Delta(b) = \left(\sum_{(a)} a_1 \otimes a_2\right)\left(\sum_{(b)} b_1 \otimes b_2\right) = \sum_{(a)} \sum_{(b)} a_1 b_1 \otimes a_2 b_2$$

Require $\Delta(ab) = \Delta(a)\Delta(b)$

II, cont.

Also require

$$\begin{tikzcd} A \otimes A \arrow[r, "m"] \arrow[d, "\varepsilon_{A\otimes A}"'] & A \arrow[d, "\varepsilon"] \\ K \arrow[r, "id"'] & K \end{tikzcd}$$

$$\begin{tikzcd} a \otimes b \arrow[r] \arrow[d] & ab \arrow[d] \\ \varepsilon(a)\varepsilon(b) \arrow[r] & \varepsilon(ab) \end{tikzcd}$$

Require

$$\varepsilon(ab) = \varepsilon(a)\,\varepsilon(b)$$

II, cont.

When is $u: K \to A$ a coalg morphism?

Require

$$\begin{tikzcd} k \arrow[r, "u"] \arrow[d] & A \arrow[d, "\Delta"] \\ k \otimes k \arrow[r, "u \otimes u"'] & A \otimes A \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1_A \arrow[d, "\Delta(1_A)"] \\ 1 \otimes 1 \arrow[r] & 1_A \otimes 1_A \end{tikzcd}$$

$$1 \downarrow 1 \otimes 1$$

require $\Delta(1_A) = 1_A \otimes 1_A$

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "id"'] & A \arrow[d, "\varepsilon"] \\ k \arrow[r, "id"'] & k \end{tikzcd}
\qquad
\begin{tikzcd} 1 \arrow[r] \arrow[d] & 1_A \arrow[d] \\ 1 \arrow[r] & 1 = \varepsilon(1_A) \end{tikzcd}$$

require $\varepsilon(1_A) = 1$

COR. Assume A is both a k-algebra and k-coalgebra. Then TFAE:

(i) $\Delta: A \to A \otimes A$ and $\varepsilon: A \to k$ are alg morphisms

(ii) $m: A \otimes A \to A$ and $u: k \to A$ are coalg morphisms

We call A a k-<u>bialgebra</u> whenever (i), (ii) hold

LEM Assume A is both a k-algebra and k-coalgebra. Then A is a bialg iff the following four diagrams commute:

$$\begin{tikzcd}
A\otimes A \arrow[rr, "m"] \arrow[d, "\Delta\otimes\Delta"'] & & A \arrow[dd, "\Delta"] \\
A\otimes A\otimes A\otimes A \arrow[d, "1\otimes\tau\otimes 1"'] & & \\
A\otimes A\otimes A\otimes A \arrow[rr, "m\otimes m"'] & & A\otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes A \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & k\otimes k \arrow[d, "1\otimes 1 \mapsto 1"] \\
A \arrow[r, "\varepsilon"'] & k
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, "u"] \arrow[d, "1 \mapsto 1\otimes 1"'] & A \arrow[d, "\Delta"] \\
k\otimes k \arrow[r, "u\otimes u"'] & A\otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[rr, "id"] \arrow[dr, "u"'] & & k \\
& A \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

pf the diagrams assert

$$\Delta(ab) = \Delta(a)\Delta(b)$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$$
$$\Delta(1_A) = 1_A\otimes 1_A$$
$$\varepsilon(1_A) = 1$$

$\square$

Investigate III

Assume A is a k-algebra

when is $u: k \to A$ an alg morph ?

Require

$$\begin{array}{ccc} u(\alpha\beta) & \overset{?}{=} & u(\alpha)\, u(\beta) \\ \| & & \| \quad\quad \| \\ (\alpha\beta)\, 1_A & & (\alpha 1_A)\,(\beta 1_A) \\ & & \| \\ & & \alpha\beta\,(1_A 1_A) \\ & & \| \\ & \text{ok} & \alpha\beta\, 1_A \end{array}$$

Require

$u(1) = 1_A$ ok

u is always an alg morph

9/12/16

11

III, cnt.

When is $m: A\otimes A \to A$ an algebra morphism?

Require

$x = a_1 \otimes b_1$

$y = a_2 \otimes b_2$

$$m(xy) = \underbrace{\underbrace{m(x)}_{=a_1 b_1}\ \underbrace{m(y)}_{=a_2 b_2}}_{a_1 b_1 a_2 b_2}$$

$$m(a_1 a_2 \otimes b_1 b_2) = a_1 a_2 b_1 b_2$$

Require $a_2 b_1 = b_1 a_2$

m is alg morphism iff A is commutative:

$$\begin{tikzcd} A\otimes A \arrow[r, "m"] \arrow[d, "\tau_{A,A}"'] & A \arrow[d, "id"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

Investigate IV

A coalgebra C is <u>cocommutative</u>

whenever

$$\begin{tikzcd} C \otimes C & C \arrow[l, "\Delta"'] \\ C \otimes C \arrow[u, "\tau_{C,C}"] & C \arrow[l, "\Delta"] \arrow[u, "id"'] \end{tikzcd}$$

In other words, for $c \in C$

$$\sum_{(c)} c_1 \otimes c_2 = \sum_{(c)} c_2 \otimes c_1$$

Assume A is coalg.

when is $\varepsilon : A \to K$ a coalg morphism?

Require

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon} & K \\ \Delta \downarrow & & \downarrow 1 \mapsto 1 \otimes 1 \\ A \otimes A & \xrightarrow[\varepsilon \otimes \varepsilon]{} & K \otimes K \end{array}$$

$$\begin{array}{ccc} a & \longrightarrow & \varepsilon(a) \\ \downarrow & & \downarrow \\ \sum_{(a)} a_1 \otimes a_2 & \longrightarrow & \sum_{(a)} \varepsilon(a_1) \otimes \varepsilon(a_2) \end{array}$$

Require

$$\varepsilon(a) = \sum_{(a)} \varepsilon(a_1)\, \varepsilon(a_2) \qquad *$$

Recall $a = \sum_{(a)} \varepsilon(a_1)\, a_2$

Apply ε to get $*$

Also require

$$\begin{tikzcd} A \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"'] & k \arrow[d, "id"] \\ k \arrow[r, "id"'] & k \end{tikzcd}$$

OK

ε is always a coalg morphism.

When is $\Delta : A \to A\otimes A$ a coalg morphism?

Require

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A\otimes A \\ \Delta \downarrow & & \downarrow \Delta_{A\otimes A} \\ A\otimes A & \xrightarrow[\Delta\otimes\Delta]{} & A\otimes A\otimes A\otimes A \end{array}$$

$$\begin{array}{ccc} a & \longrightarrow & \sum_{(a)} a_1\otimes a_2 \\ & & \downarrow \\ \downarrow & & \sum_{(a)} a_1\otimes a_3\otimes a_2\otimes a_4 \\ \sum_{(a)} a_1\otimes a_2 & \longrightarrow & \sum_{(a)} a_1\otimes a_2\otimes a_3\otimes a_4 \end{array}$$

ok if A is co-com

Also require

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A\otimes A \\ \varepsilon \downarrow & & \downarrow \varepsilon_{A\otimes A} \\ k & \xrightarrow[id]{} & k \end{array}$$

$$\begin{array}{ccc} a & \longrightarrow & \sum_{(a)} a_1\otimes a_2 \\ \downarrow & & \downarrow \\ \varepsilon(a) & \longrightarrow & \sum_{(a)} \varepsilon(a_1)\varepsilon(a_2) = \varepsilon(a) \end{array}$$

ok

Δ is a coalg morphism if A is co-commutative.

Summary Given A, algebra or coalgebra

when is: (if A alg) (if A coalg)

		alg morphism	coalg morphism
(if A alg)	m	iff A is com	iff $\Delta(ab) = \Delta(a)\Delta(b)$ $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$
	u	yes	iff $\Delta(1_A) = 1_A \otimes 1_A$ $\varepsilon(1_A) = 1$
(if A coalg)	Δ	iff $\Delta(ab) = \Delta(a)\Delta(b)$ $\Delta(1_A) = 1_A \otimes 1_A$	if A is cocom
	ε	iff $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ $\varepsilon(1_A) = 1$	yes