

Lec 3 Monday Sept 12

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We continue to discuss coalgebras.

Recall: Given coalgebras C, C'

then $C \otimes C'$ is a coalgebra with

$$\Delta : C \otimes C' \xrightarrow{\Delta \otimes \Delta'} C \otimes C \otimes C' \otimes C' \xrightarrow{1 \otimes \tau \otimes 1} C \otimes C' \otimes C \otimes C'$$

$$\varepsilon : C \otimes C' \xrightarrow{\varepsilon \otimes \varepsilon'} k \otimes k \xrightarrow{1 \otimes 1} k$$

Def Given coalgebras C, C'

A map $\varphi: C \rightarrow C'$ is a coalgebra morphism

whenever φ is a K -module hom that makes

these diagrams commute:

$$\begin{array}{ccc}
 C & \xrightarrow{\varphi} & C' \\
 \Delta \downarrow & & \downarrow \Delta' \\
 C \otimes C & \xrightarrow{\varphi \otimes \varphi} & C' \otimes C'
 \end{array}$$

$$\begin{array}{ccc}
 C & \xrightarrow{\varphi} & C' \\
 \varepsilon \downarrow & & \downarrow \varepsilon' \\
 K & \xrightarrow{id} & K
 \end{array}$$

Given A , an algebra or coalgebra

When is f (if A is alg) alg morphism (if A is coalg) coalg morphism

(if A is alg)

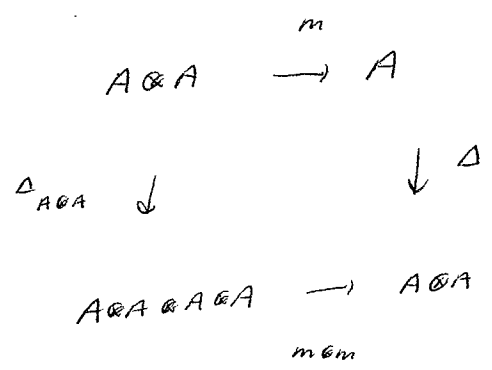
(if A is coalg)

		alg morphism	coalg morphism
m			
u			
			III II
			I IV
Δ	require $\Delta(ab) = \Delta(a)\Delta(b)$ $\Delta(1_A) = 1_A \otimes 1_A$		
ε	require $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ $\varepsilon(1_A) = 1$		

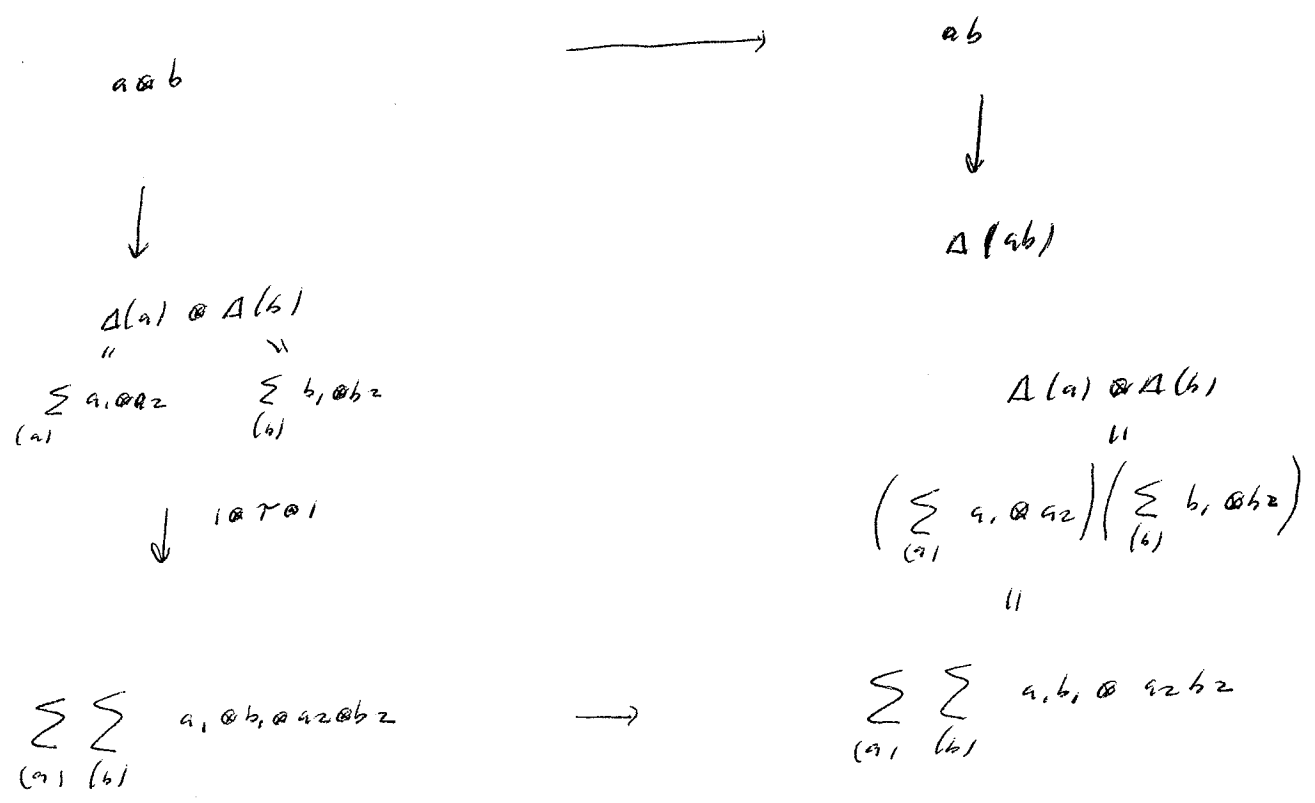
Investigate II

When is $m: A \otimes A \rightarrow A$ a coalg morphism?

Require



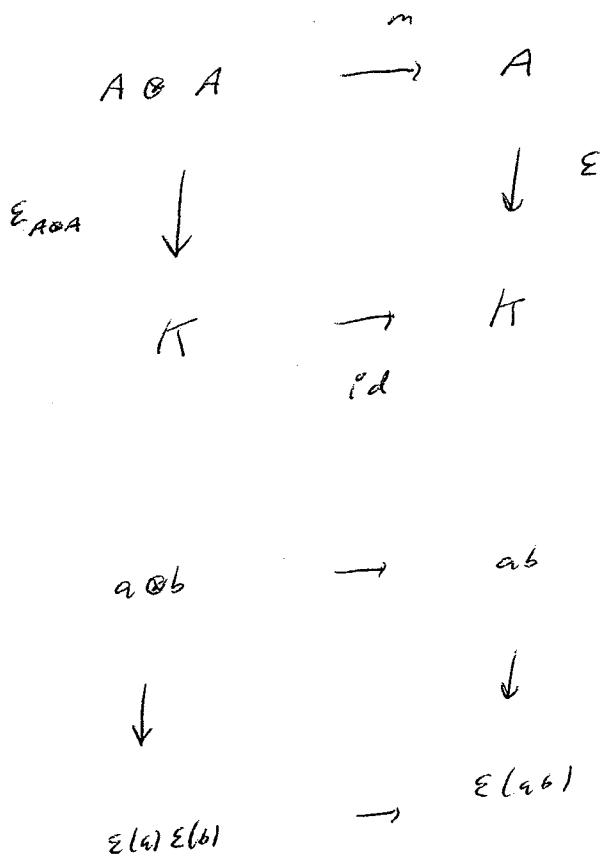
For $a, b \in A$



Require $\Delta(ab) = \Delta(a) \otimes \Delta(b)$

II, cont.

Also require



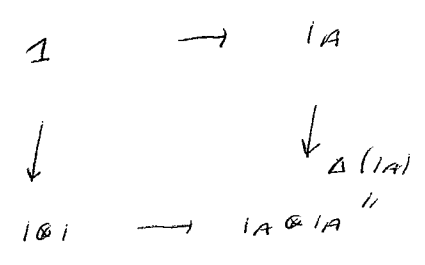
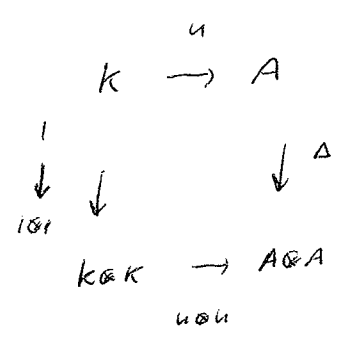
Require

$$\varepsilon(ab) = \varepsilon(a) \varepsilon(b)$$

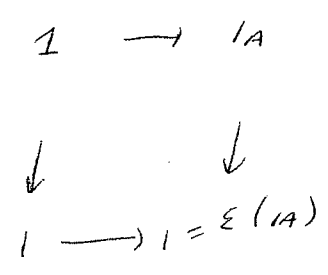
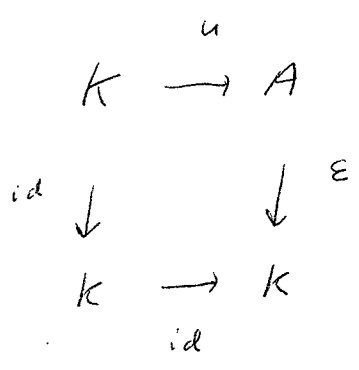
II, cont.

When is $u: K \rightarrow A$ a coalg morphism?

Require



require $\Delta(1_A) = 1_A \otimes 1_A$



require $\epsilon(1_A) = 1$

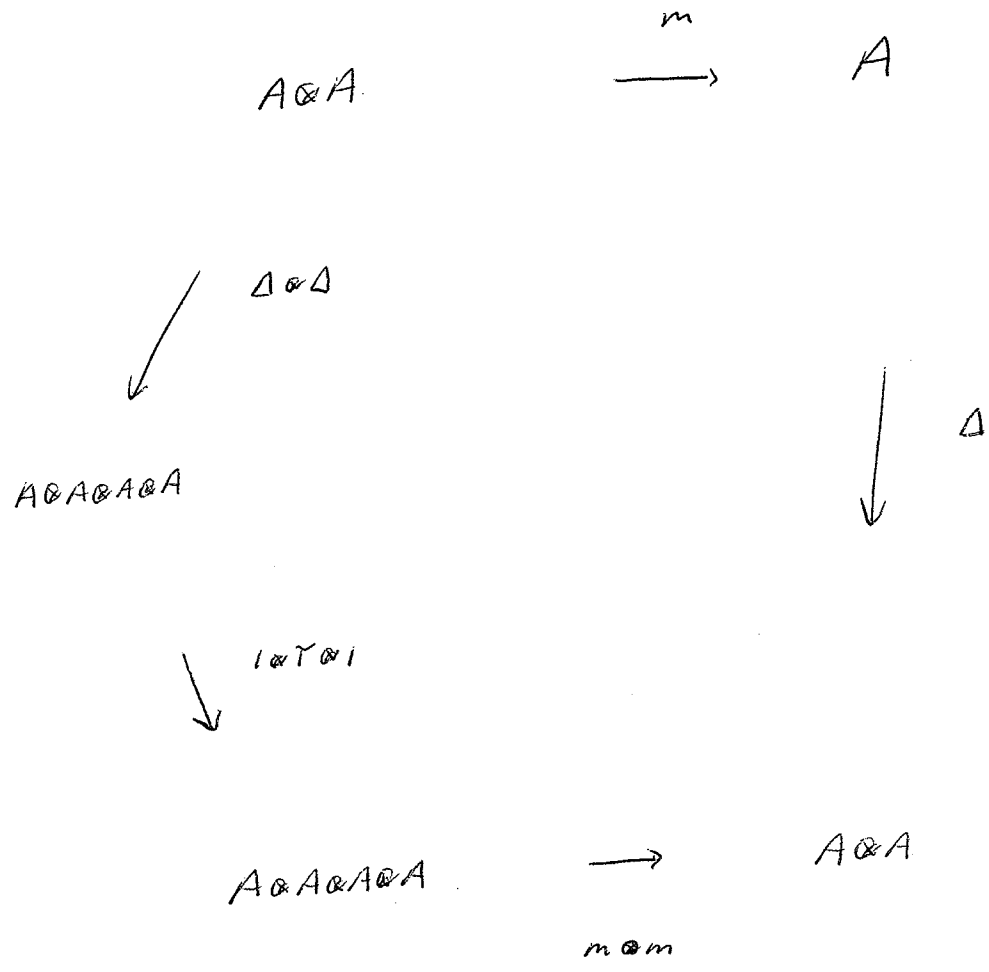
COR. Assume A is both a k -algebra and k -coalgebra. Then TFAE:

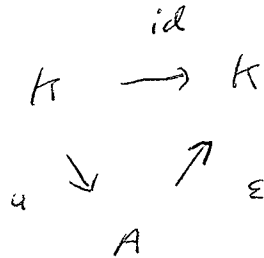
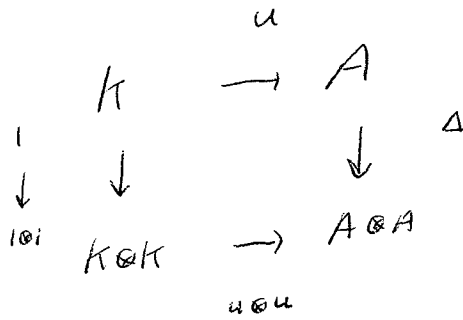
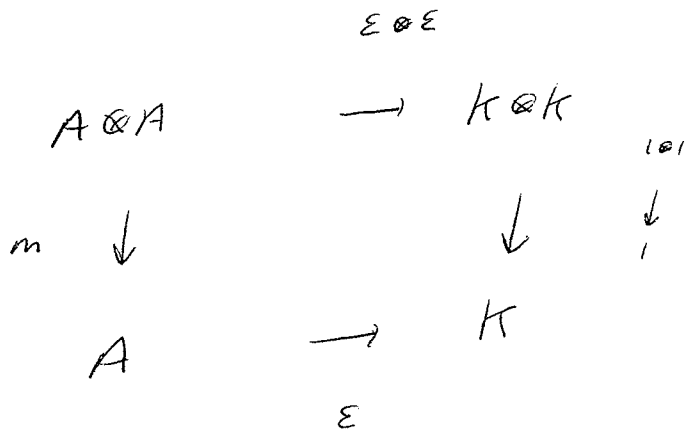
(i) $\Delta: A \rightarrow A \otimes A$ and $\varepsilon: A \rightarrow k$ are alg morphisms

(ii) $m: A \otimes A \rightarrow A$ and $u: k \rightarrow A$ are coalg morphisms

We call A a k -bialgebra whenever (i), (ii) hold

LEM Assume A is both a k -algebra and k -coalgebra. $\textcircled{9}$
Then A is a bialg iff the following four diagrams commute:





pf the diagrams assert

$$\Delta(ab) = \Delta(a) \Delta(b)$$

$$\varepsilon(ab) = \varepsilon(a) \varepsilon(b)$$

$$\Delta(1_A) = 1_A \otimes 1_A$$

$$\varepsilon(1_A) = 1$$

□

Investigate III

Assume A is a k -algebra

When is $u: k \rightarrow A$ an alg morph?

Require $u(\alpha\beta) \stackrel{?}{=} u(\alpha)u(\beta)$

\parallel
 $(\alpha 1_A) \quad (\beta 1_A)$
 \parallel
 $(\alpha\beta) 1_A$
 \parallel
 $\alpha\beta (1_A 1_A)$
 \parallel
 $\alpha\beta 1_A$

OK

Require $u(1) = 1_A$ OK

u is always an alg morph

III, cont.

When is $m: A \otimes A \rightarrow A$ an algebra morphism? 9/12/16
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Require

$$\begin{aligned} m(xy) &= \underbrace{m(x)}_{a_1 b_1} \underbrace{m(y)}_{a_2 b_2} \\ &= \underbrace{a_1 a_2}_{a_1 a_2} \underbrace{b_1 b_2}_{b_1 b_2} \\ &= a_1 a_2 b_1 b_2 \end{aligned}$$

$x = a_1 \otimes b_1$
 $y = a_2 \otimes b_2$

Require $a_2 b_1 = b_1 a_2$

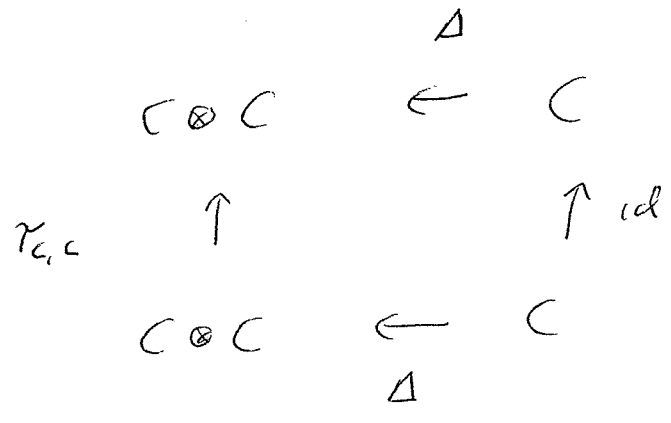
m is alg morphism iff A is commutative:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ \tau_{A,A} \downarrow & & \downarrow \text{id} \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

Investigate IV

A coalgebra C is cocommutative

whenever



In other words, $\forall c \in C$

$$\sum_{(c)} c_1 \otimes c_2 = \sum_{(c)} c_2 \otimes c_1$$

Assume A is coalg.

When is $\epsilon: A \rightarrow K$ a coalg morphism?

Require

$$\begin{array}{ccc}
 & \epsilon & \\
 A & \longrightarrow & K \\
 \Delta \downarrow & & \downarrow \downarrow \\
 & & (a) \\
 A \otimes A & \xrightarrow{\epsilon \otimes \epsilon} & K \otimes K
 \end{array}$$

$$\begin{array}{ccc}
 a & \longrightarrow & \epsilon(a) \\
 \downarrow & & \downarrow \\
 \sum_{(a)} a_1 \otimes a_2 & \longrightarrow & \sum_{(a)} \epsilon(a_1) \otimes \epsilon(a_2)
 \end{array}$$

Require

$$\epsilon(a) = \sum_{(a)} \epsilon(a_1) \epsilon(a_2)$$



Recall

$$a = \sum_{(a)} \epsilon(a_1) a_2$$

Apply ϵ to get *

IV, cont

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Also require

$$\begin{array}{ccc} & \varepsilon & \\ & \longrightarrow & \\ A & & k \\ & \downarrow & \downarrow \text{id} \\ \varepsilon & & \\ k & \longrightarrow & k \\ & \text{id} & \end{array}$$

OK

ε is always a coalg morphism.

IV, cont.

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When is $\Delta : A \rightarrow A \otimes A$ a coalg morphism?

Require

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 \Delta \downarrow & & \downarrow \Delta_{A \otimes A} \\
 A \otimes A & \xrightarrow{\Delta_{A \otimes A}} & A \otimes A \otimes A \otimes A \\
 & \Delta_{A \otimes A} &
 \end{array}$$

$$\begin{array}{ccc}
 a & \longrightarrow & \sum_{(a_i)} a_1 \otimes a_2 \\
 \downarrow & & \downarrow \\
 \sum_{(a_i)} a_1 \otimes a_2 & \longrightarrow & \sum_{(a_i)} a_1 \otimes a_2 \otimes a_3 \otimes a_4 \\
 & & \downarrow \\
 & & \sum_{(a_i)} a_1 \otimes a_2 \otimes a_3 \otimes a_4
 \end{array}$$

ok if A is co-com

Also require

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 \varepsilon \downarrow & & \downarrow \varepsilon_{A \otimes A} \\
 k & \xrightarrow{id} & k
 \end{array}$$

$$\begin{array}{ccc}
 a & \longrightarrow & \sum_{(a_i)} a_1 \otimes a_2 \\
 \downarrow & & \downarrow \\
 \varepsilon(a) & \longrightarrow & \sum_{(a_i)} \varepsilon(a_1) \varepsilon(a_2) = \varepsilon(a)
 \end{array}$$

OK

Δ is a coalg morphism if A is co-commutative.

Summary

Given A , algebra or coalgebra

when is: (if A alg) (if A coalg)

	alg morphism	coalg morphism
m (if A alg)	iff A is com	iff $\Delta(ab) = \Delta(a)\Delta(b)$ $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$
η	yes	iff $\Delta(1_A) = 1_A \otimes 1_A$ $\varepsilon(1_A) = 1$
Δ (if A coalg)	iff $\Delta(ab) = \Delta(a)\Delta(b)$ $\Delta(1_A) = 1_A \otimes 1_A$	if A is cocom
ε	iff $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ $\varepsilon(1_A) = 1$	yes