Lec 29 Friday Nov 11

11/11/16

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For the sym functions Λ we identify

$\Lambda^0 = \Lambda$ via the Hall $\langle , \rangle$

LEM For $f, g \in \Lambda$ and $n \in \mathbb{N}$

$$f \cdot \left(h_n^{\perp}(g)\right) = \qquad *$$

$$\sum_{k=0}^{n} (-1)^k \, h_{n-k}^{\perp}\left(\left(e_k^{\perp}(f)\right) \cdot g\right)$$

pf Eval RHS of $*$

Recall

$$\Delta(h_r) = \sum_{i=0}^{r} h_i \otimes h_{r-i}$$

Observe

RHS =

$$\sum_{k=0}^{n} (-1)^k \sum_{j=0}^{n-k} \underbrace{\left(h_j^\perp\left(e_k^\perp(f)\right)\right)}_{\substack{\shortparallel \\ (h_j e_k)^\perp (f)}} \cdot \left(h_{n-k-j}^\perp(g)\right)$$

$$= \sum_{l=0}^{n} \sum_{\substack{j,k\geq 0 \\ j+k=l}} (-1)^k \left((h_j e_k)^\perp (f)\right) \cdot \left(h_{n-l}^\perp(g)\right)$$

$$= \sum_{l=0}^{n} \left(\Big(\underbrace{\sum_{\substack{j,k\geq 0 \\ j+k=l}} (-1)^k h_j e_k}_{\substack{\shortparallel \\ \begin{cases} 0 & \text{if } l\neq 0 \\ 1 & \text{if } l=0 \end{cases}}}\Big)^\perp (f)\right) \cdot h_{n-l}^\perp(g)$$

$$= \underbrace{1^\perp(f)}_{\substack{\shortparallel \\ f}} \cdot h_n^\perp(g)$$

$$= f \cdot h_n^\perp(g)$$

$\square$

thm (skew Pieri rule I)

Fix $\lambda, \mu \in Par$ and $n \in \mathbb{N}$

$$h_n s_{\lambda/\mu} = \sum_{\substack{\lambda^+, \mu^- \in Par \\ \lambda^+/\lambda \text{ is horiz strip} \\ \mu/\mu^- \text{ is vert strip} \\ |\lambda^+/\lambda| + |\mu/\mu^-| = n}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-}$$

pf Fix $g \in \Lambda$ show

$$(LHS, g) = (RHS, g)$$

obs

$$(LHS, g) = (h_n s_{\lambda/\mu}, g)$$

$$= \left(h_n \cdot \left(s_\mu^\perp(s_\lambda)\right), g\right)$$

$$= \left(s_\mu^\perp(s_\lambda), h_n^\perp(g)\right)$$

$$= \left(s_\lambda, \underbrace{s_\mu \cdot \left(h_n^\perp(g)\right)}_{\overset{\shortparallel}{f}}\right)$$

[apply prev LEM]

$$= \sum_{k=0}^{n} (-1)^k \left(s_\lambda, h_{n-k}^\perp\left(e_k^\perp(s_\mu) \cdot g\right)\right)$$

$$= \sum_{k=0}^{n} (-1)^k \left(\underbrace{h_{n-k} \cdot s_\lambda}_{\substack{\shortparallel \\ \sum\limits_{\substack{\lambda^+ \in Par \\ \lambda^+/\lambda \text{ is horiz} \\ (n-k)\text{-strip}}} s_{\lambda^+}}}, \underbrace{\left(e_k^\perp(s_\mu)\right)}_{\substack{\shortparallel \\ \sum\limits_{\substack{\mu^- \in Par \\ \mu/\mu^- \text{ is vert} \\ k\text{-strip}}} s_{\mu^-}}} \cdot g \right)$$

$$= \sum_k \sum_{\lambda^+} \sum_{\mu^-} (-1)^k \underbrace{\left(s_{\lambda^+}, s_{\mu^-} \cdot g\right)}_{\substack{\shortparallel \\ \left(\underbrace{s_{\mu^-}^\perp(s_{\lambda^+})}_{\shortparallel\ s_{\lambda^+/\mu^-}}, g\right)}}$$

$$= (RHS, g)$$

$\square$

thm (skew Pieri Rule II)

For $\lambda, \mu \in Par$ and $n \in \mathbb{N}$

$$e_n \, s_{\lambda/\mu} = \sum_{\substack{\lambda^+, \mu^- \in Par \\ \lambda^+/\lambda \text{ is vert strip} \\ \mu/\mu^- \text{ is horiz strip} \\ |\lambda^+/\lambda| + |\mu/\mu^-| = n}} (-1)^{|\mu/\mu^-|} \, s_{\lambda^+/\mu^-}$$

pf Apply fund invol ω to everything in RULE I $\square$

Some determinant formula for e_n, p_n

Recall that for $n \in \mathbb{N}$

$$(n+1)e_{n+1} = \sum_{i=0}^{n} (-1)^{n-i} e_i \, p_{n-i+1} \qquad \star$$

LEM For $n \geq 0$

$$(n+1)!\, e_{n+1} = \begin{vmatrix} p_1 & 1 & & & & \\ p_2 & p_1 & 2 & & 0 & \\ \vdots & p_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & p_1 & n \\ p_{n+1} & \cdots & & & p_2 & p_1 \end{vmatrix} = E_{n+1}$$

pf By ind on n

$n = 0$ $e_1 = p_1$ ✓

$n \geq 1$ Mult both sides of $\star$ by $n!$

$$\underbrace{(n+1)!\,e_{n+1}}_{\overset{?}{=}\ |E_{n+1}|} = \underbrace{\sum_{i=0}^{n} (-1)^{n-i} \underbrace{i!\,e_i}_{=\ |E_i|} \; p_{n-i+1} \; n(n-1)\cdots(i+1)}_{=\ \text{det expansion of } E_{n+1} \text{ along bottom row}}$$

$\square$

LEM $\forall n \geq 0$

$$p_{n+1} = \underbrace{\begin{vmatrix} e_1 & 1 & & & & & \\ 2e_2 & e_1 & 1 & & & & \\ 3e_3 & e_2 & e_1 & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ ne_n & & & & & & 1 \\ (n+1)e_{n+1} & e_n & \cdots & & e_2 & e_1 & 1 \end{vmatrix}}_{= |P_{n+1}|}$$

pf Use ind on n

$n=0$ $\quad p_1 = e_1$ ✓

$n \geq 1$

By $*$

$$(n+1)e_{n+1} - (-1)^n p_{n+1} = \sum_{i=1}^{n} (-1)^{n-i} e_i \, p_{n-i+1}$$

Expanding $|P_{n+1}|$ along bottom row

$$(n+1)e_{n+1} - (-1)^n |P_{n+1}| = \sum_{i=1}^{n} (-1)^{n-i} e_i \underbrace{|P_{n-i+1}|}_{= p_{n-i+1} \text{ ind}}$$

So $\quad p_{n+1} = |P_{n+1}|$

$\square$

Aside on the Littlewood-Richardson coef $C^{\lambda}_{\mu\nu}$

Assume $\mathbb{K} = \mathbb{Z}$

Let $\mathbb{F}$ = field, let $n \in \mathbb{N}$

Let V = vector space over $\mathbb{F}$ with dim n.

Consider a nilpotent $\mathbb{F}$-linear map $A : V \to V$

Put A in Jordan normal form:

$\exists$ basis for V wrt which the matrix rep A is

$$A : \begin{array}{c} \left(\begin{array}{c|c|c|c} \begin{smallmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{smallmatrix} & 0 & 0 & \cdots \\ \hline 0 & \begin{smallmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{smallmatrix} & & \\ \hline & & \ddots & \\ \hline 0 & & & \begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{smallmatrix} \\ \vdots & & & \end{array} \right) \\ \lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_\ell \end{array} \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_\ell \end{array}$$

Jordan normal form is uniquely det by $\lambda_1, \lambda_2, \ldots, \lambda_\ell$
(up to perm)

WLOG $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$

so $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of n.

Call this partition the <u>Jordan type</u> of A.

Describe the Jordan type λ of A

<u>Case</u> A has single Jordan block

$$\lambda = (n)$$

$$A: \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & 0 \end{pmatrix}$$

For $0 \le i \le n$

$$\dim(\ker(A^i)) = i$$

Write $\mu = \lambda^t = (1, 1, \dots 1)$

$$\dim(\ker(A^i)) = \mu_1 + \mu_2 + \dots + \mu_i \qquad (0 \le i \le n)$$

For any number of Jordan blocks,

$$\dim(\ker(A^i)) = \mu_1 + \mu_2 + \dots + \mu_i \qquad (0 \le i \le n)$$

where μ = transpose of the Jordan type of A (ex)

So for $1 \le i \le n$

$$\mu_i = \dim(\ker(A^i)) - \dim(\ker(A^{i-1}))$$

Pick any subspace $U \subseteq V$ that is A-inv

Let $r = \dim U$

The restriction $A/_U$ is nilpotent; let μ denote its Jordan type. So $\mu \in Par_r$.

Consider the quotient vectorspace V/U.

A induces an $\mathbb{F}$-linear map

$$V/U \to V/U$$
$$v + U \to A(v) + U$$

This map is nilpotent; let ν denote its Jordan type.

So $\nu \in Par_{n-r}$

So $|\mu| + |\nu| = |\lambda| = n$

<u>Thm</u> For $\lambda, \mu, \nu \in Par$ these come from above construction iff $c^{\lambda}_{\mu\nu} \neq 0$

(See ex 2.99 in notes)