Lecture 28 Wed Nov 9

Recall that for $\lambda, \mu, \nu \in Par$

$c^{\lambda}_{\mu\nu}$ = # of col strict tableaux T

that have shp λ/μ and content ν st

cont($T|_{cols \geq j}$) is partition $\forall j \geq 1$

"Littlewood-Richardson rule"

— o —

A skew shape is called a __horizontal n-strip__

whenever it has exactly n boxes, no two in same column

A vertical n-strip is sim defined.

__Thm__ For $n \in \mathbb{N}$ and $\mu \in Par$

(i) $$s_\mu h_n = \sum_{\substack{\lambda \in Par \\ \lambda/\mu \text{ is horiz } n\text{-strip}}} s_\lambda$$ "Pieri Rule"

(ii) $$s_\mu e_n = \sum_{\substack{\lambda \in Par \\ \lambda/\mu \text{ is vert } n\text{-strip}}} s_\lambda$$

pf (i) Recall

$$h_n = s_\nu \qquad \nu = (n)$$

$$s_\mu s_\nu = \sum_{\lambda \in Par} c^\lambda_{\mu\nu} s_\lambda$$

For $\lambda \in Par$ show

$$c^\lambda_{\mu\nu} = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is horiz } n\text{-strip} \\ 0 & \text{else} \end{cases}$$

First assume $c^\lambda_{\mu\nu} \neq 0$ so $\exists T$ that satisfies LR rule

Since $\nu = (n)$ T has n boxes, each contains 1

Since T is col strict, no 2 boxes in same col

So λ/μ is horiz n-strip

Conversely assume λ/μ is horiz n-strip.

The tableaux T of shp λ/μ with all boxes 1

is unique tableaux that satisfies LR rule.

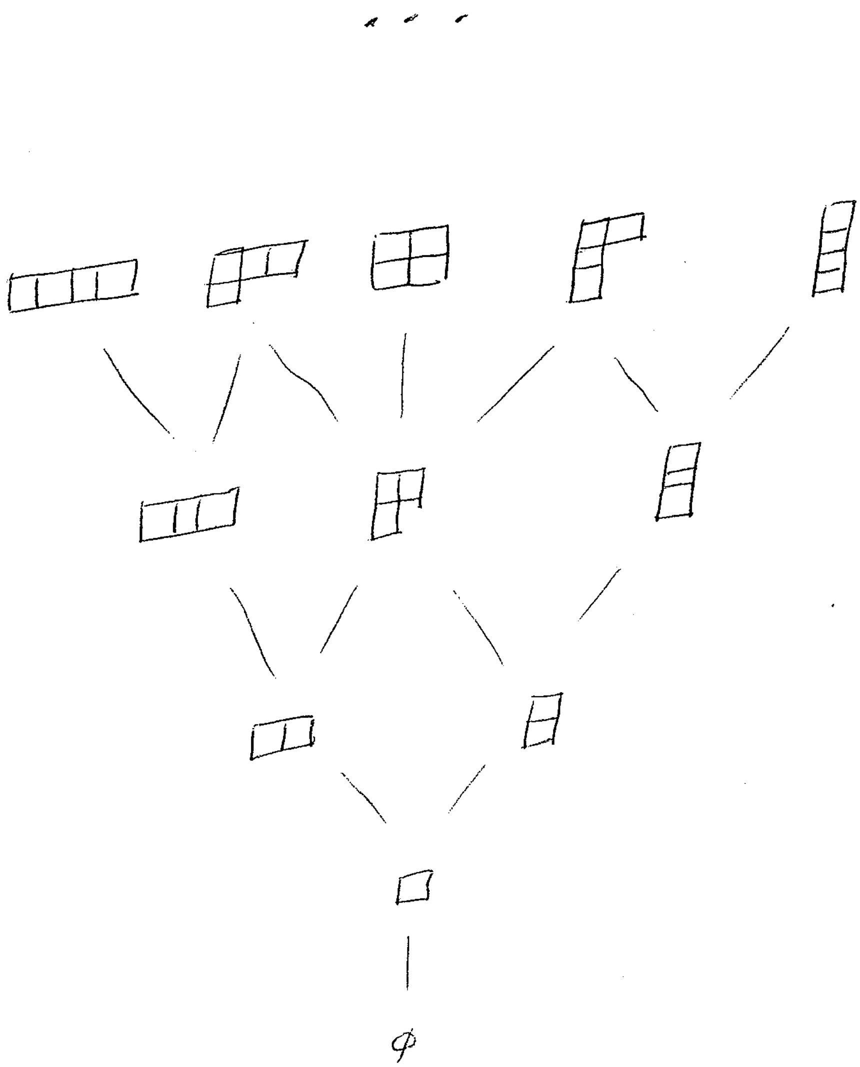
So $c^\lambda_{\mu\nu} = 1$

(ii) Apply fund invol ω to each side of (i)

In above thm next consider $n=1$

View Par as a poset with partial order $\subseteq$.

Hasse diagram is



"Young's lattice"

For $\lambda, \mu \in Par$

λ covers μ in Young's lattice

iff

λ/μ is single box

(ie horiz 1-strip ∩ vert 1-strip)

Cor For $\mu \in Par$

$$s_\mu h_1 = \sum_{\substack{\lambda \in Par \\ \lambda \text{ covers } \mu}} s_\lambda$$

Define

$$\mathcal{U} : \Lambda \to \Lambda$$
$$f \to h_1 f$$

For $\mu \in Par$

$$\mathcal{U}(s_\mu) = \sum_{\substack{\lambda \in Par \\ \lambda \text{ covers } \mu}} s_\lambda$$

"raising map"

Recall Hall inner prod $(,)$ on Λ

Schur functions are orthonormal wrt $(,)$

Define $D = U^* =$ adjoint wrt $(,)$

then for $\mu \in Par$

$$D(s_\mu) = \sum_{\substack{\lambda \in Par \\ \mu \text{ covers } \lambda}} s_\lambda$$

"lowering map"

Remark

$$DU - UD = id$$

— o —

Next goal: to prove

<u>Thm</u> For $\lambda, \mu \in Par$ and $n \in \mathbb{N}$

$$\text{(i)} \quad s_{\lambda/\mu} h_n = \sum_{\substack{\lambda^+, \mu^- \in Par \\ \lambda^+/\lambda \text{ is horiz strip} \\ \mu/\mu^- \text{ is vert strip} \\ |\lambda^+/\lambda| + |\mu/\mu^-| = n}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-}$$

"Skew Pieri rule"

(ii)

$$s_{\lambda/\mu}\, e_n = \sum_{\substack{\lambda^+, \mu^- \in \mathrm{Par} \\ \lambda^+/\lambda \text{ is vert strip} \\ \mu/\mu^- \text{ is horiz strip} \\ |\lambda^+/\lambda| + |\mu/\mu^-| = n}} (-1)^{|\mu/\mu^-|}\, s_{\lambda^+/\mu^-}$$

Def For $\mu \in \mathrm{Par}$ define the k-module hom

$$s_\mu^\perp : \Lambda \longrightarrow \Lambda$$
$$s_\lambda \longmapsto s_{\lambda/\mu}$$

where we recall $s_{\lambda/\mu} = 0$ if $\mu \not\subseteq \lambda$

Remark For $\mu = (1)$

$$s_\mu^\perp = D$$

Aside on Hopf algebras

Until further notice, let H = graded Hopf alg of finite type, with graded dual H^0

Recall the k-bilin form

$$(\, , \,) \qquad \begin{array}{ccc} H^0 \times H & \to & k \\ f \quad a & \to & f(a) \end{array}$$

Def With above notation, $\forall f \in H^0$ define the k-module hom

$$f^\perp : \quad \begin{array}{ccc} H & \to & H \\ a & \to & \sum_{(a)} f(a_1) a_2 \end{array}$$

Ex For $H = \Lambda$, identify $\Lambda^{\circ} = \Lambda$ via Hall $(\,,\,)$

Pick $\mu \in \mathrm{Par}$ and let $f = s_\mu$

then $f^{\perp}(s_\lambda) = s_{\lambda/\mu}$ $\forall \lambda \in \mathrm{Par}$

Pf Recall

$$\Delta(s_\lambda) = \sum_{\rho,\nu} c^{\lambda}_{\rho\nu}\, s_\rho \otimes s_\nu$$

So

$$f^{\perp}(s_\lambda) = \sum_{\rho,\nu} c^{\lambda}_{\rho\nu} \underbrace{(\overset{= s_\mu}{f}, s_\rho)}_{= \delta_{\rho,\mu}} s_\nu$$

$$= \sum_{\nu} c^{\lambda}_{\mu\nu}\, s_\nu$$

$$= s_{\lambda/\mu}$$

□

Back to general H, for $f \in H^0$ compare

these maps:

$$f^{\perp}: \quad H \to H \qquad \ast$$
$$a \to \sum_{(a)} f(a_1)\, a_2$$

$$H^0 \to H^0 \qquad \ast\ast$$
$$g \to f \star g$$
$$\uparrow$$
mult in alg H^0

LEM The maps $\ast$, $\ast\ast$ are adjoint wrt $(\ ,\ )$

pf For $a \in H$ and $g \in H^0$ show

$$(g, f^{\perp} a) \overset{?}{=} (f \star g, a)$$

$$(g, f^{\perp} a) = g\Big(\sum_{(a)} f(a_1)\, a_2\Big) = \sum_{(a)} f(a_1)\, g(a_2)$$

$$(f \star g, a) = \sum_{(a)} f(a_1)\, g(a_2)$$

ok $\square$

LEM Fa $f, g \in H^\circ$

the following maps coincide:

$$(f * g)^\perp : \quad H \to H \qquad *$$

$$H \xrightarrow{f^\perp} H \xrightarrow{g^\perp} H \qquad **$$

pf Fa $a \in H$

$$(f * g)^\perp(a) = \sum_{(a)} (f * g)(a_1)\, a_2$$

$$= \sum_{(a)} f(a_1)\, g(a_2)\, a_3$$

Also

$$g^\perp(f^\perp(a)) = g^\perp\Big(\sum_{(a)} f(a_1)\, a_2 \Big)$$

$$= \sum_{(a)} f(a_1)\, g^\perp(a_2)$$

$$= \sum_{(a)} f(a_1)\, g(a_2)\, a_3$$

ok

$\square$

<u>Note</u> By above LEM

H becomes a right H°-module.

LEM For $f \in H^\circ$ and $a, b \in H$

$$f^\perp(ab) = \sum_{(f)} f_1^\perp(a)\, f_2^\perp(b)$$

pf LHS: Recall

$$\Delta(ab) = \Delta(a)\Delta(b) = \sum_{(a)}\sum_{(b)} a_1 b_1 \otimes a_2 b_2$$

So

$$f^\perp(ab) = \sum_{(a)}\sum_{(b)} f(a_1 b_1)\, a_2 b_2 \qquad \maltese$$

Recall comod Δ of H° satisfies

$$\Delta(f) = \sum_{(f)} f_1 \otimes f_2$$

with

$$f(ab) = \sum_{(f)} f_1(a)\, f_2(b)$$

So $\maltese$ becomes

$$f^\perp(ab) = \sum_{(a)}\sum_{(b)}\sum_{(f)} f_1(a_1)\, f_2(b_1)\, a_2 b_2$$

RHS:

$$\sum_{(f)} f_1^{\perp}(a) \; f_2^{\perp}(b) = \sum_{(f)} \sum_{(a)} \sum_{(b)} f_1(a_1) a_2 \; f_2(b_1) b_2 \quad c/_T$$

$$= \sum_{(a)} \sum_{(b)} \sum_{(f)} f_1(a_1) f_2(b_1) \, a_2 b_2$$

ok

$\square$

Recall the counit ε of H is the mult identity of alg H°

LEM For $a \in H$

$$\varepsilon^{\perp}(a) = a$$

pf $\varepsilon^{\perp}(a) = \sum_{(a)} \varepsilon(a_1) a_2 = a$ $\square$

LEM For $f \in H^\circ$ that is primitive:

$$\Delta(f) = f \otimes \varepsilon + \varepsilon \otimes f.$$

Then

$$f^\perp(ab) = f(a)b + a f(b) \qquad \forall a, b \in H$$

In other words $f^\perp$ acts on H as a derivation.

pf By prev two lemmas. $\square$