Lec 27 Monday Nov 7

Recall the Littlewood–Richardson coefs $c^\lambda_{\mu\nu}$:

$$\Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu}\, s_\mu \otimes s_\nu$$

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda$$

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu}\, s_\nu$$

Next general goal: what do the nonneg integers $c^\lambda_{\mu\nu}$ count?

Recall for $n \in \mathbb{N}$

$$s_\lambda(x_1, x_2, \dots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho} \qquad \lambda \in \mathrm{Par},\ \ell(\lambda) \le n$$

We next generalize this to include skew shapes λ/μ

<u>Notation</u> Until further notice, all partitions discussed are assumed to have $\le n$ parts. All sym functions are in the vars $x_1, x_2, \dots, x_n$. For all tableaux all entries are in $\{1, 2, \dots, n\}$

For a tableaux T and $j \ge 1$

$T|_{\text{cols} \ge j}$ means the tableaux obtained from T by deleting cols $1, 2, \dots, j-1$

$T|_{\text{cols} \le j}$ is similarly defined.

Thm For $\lambda, \mu, \nu \in Par$ with $\mu \subseteq \lambda$

$$a_{\nu+\rho}\, s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{T} a_{\nu + cont(T) + \rho}$$

sum over all col-strict tableaux T with entries in $\{1, 2, \ldots, n\}$ of shp λ/μ st $\nu + cont(T/cols \geq j)$ is a partition $\forall j \geq 1$

pf We have

$$a_{\nu+\rho}\, s_{\lambda/\mu} = \sum_{\sigma \in S_n} sgn(\sigma)\, \sigma(x^{\nu+\rho})\, \underbrace{s_{\lambda/\mu}}_{= \sigma(s_{\lambda/\mu})}$$

$a_{\nu+\rho} = \left| x_i^{\nu_j + n - j} \right|_{1 \leq i,j \leq n}$, $s_{\lambda/\mu} = s_{\lambda/\mu}(x_1, \ldots, x_n)$

$$= \sum_{\sigma \in S_n} sgn(\sigma)\, \sigma(x^{\nu+\rho}) \sum_{\substack{T = \text{col str} \\ \text{shp } \lambda/\mu}} \sigma\left(x^{cont(T)} \right)$$

$$= \sum_{\substack{T = \text{col str} \\ \text{shp } \lambda/\mu}} \sum_{\sigma \in S_n} sgn(\sigma)\, \sigma\left(x^{\nu + cont(T) + \rho} \right)$$

$$= \sum_{\substack{T = \text{col str} \\ \text{shp } \lambda/\mu}} a_{\nu + cont(T) + \rho}$$

$\not{X}$

In $*$ show all summands cancel except for those T in thm statement.

Given T = col strict tableaux shp λ/μ s.t. fails the condition in thm statement.

Consider max'l j s.t.

$$\nu + \text{cont}(T|_{\text{cols} \geq j}) \qquad ☆$$

is not a partition.

(this j exists since for suf large j $T|_{\text{cols} \geq j} = \phi$ and ν is a partition)

Given j, find minimal k s.t.

$$\text{part } k \text{ of } ☆ < \text{part } k{+}1 \text{ of } ☆$$

Since j is maximal,

$$\nu + \text{cont}(T|_{\text{cols} \geq j+1}) \qquad ☆☆$$

is a partition.

So

part $k+1$ of $\star\star$ $\leq$ part k of $\star\star$

Let

$x =$ # boxes in column j of T that contain k

$y =$ " " -- $k+1$

Since T is col strict,

$$x, y \in \{0, 1\}$$

By const

(part $k+1$ of $\star$) $-1 \geq$ part k of $\star$ $=$ part k of $\star\star$ $+ x$

part k of $\star\star$ $\geq$ part $k+1$ of $\star\star$ $=$ part $k+1$ of $\star$ $- y$

Add these inequalities and cancel

$$-1 \geq x - y$$

$$y \geq x+1$$

So $y = 1, \quad x = 0$

Now $\quad$ part k of ☆ $=$ (part $k+1$ of ☆) -1

Consider $\quad$ ☆☆☆

$$\nu + \mathrm{cont}(T/\text{cols} \geq j) + \rho$$

Obs

part k of ☆☆☆ $=$ part $k+1$ of ☆☆☆

Consider transposition $(k, k+1) \in S_n$ that swaps $k \leftrightarrow k+1$
$\quad\quad \overset{\parallel}{\sigma}$

This σ fixes ☆☆☆.

Back when we showed the Schur functions are in Λ, we employed an invol on tableaux called Bender/Knuth.

Create a new tableau T^* from T by applying B/K invol for the letters $k, k+1$, but only for cols $1, 2, \ldots, j-1$.

Leave cols $j, j+1, \ldots$ of T alone.

So $\quad T/\text{cols} \geq j \;=\; T^*/\text{cols} \geq j$

T^* is still col strict since col j of T has no box containing k

The B/K algorithm affects the content as follows

$$\theta\, \mathrm{cont}(T|_{\mathrm{cols} \le j-1}) = \mathrm{cont}(T^*|_{\mathrm{cols} \le j-1})$$

<u>claim</u>

$$\underbrace{\theta\Big(\nu + \overbrace{\mathrm{cont}(T)}^{\mathrm{cont}(T|_{\mathrm{cols}\ge j}) + \mathrm{cont}(T|_{\mathrm{cols}\le j-1})} + \rho\Big)}_{} \overset{?}{=} \nu + \underbrace{\mathrm{cont}(T^*)}_{\mathrm{cont}(T^*|_{\mathrm{cols}\ge j}) + \mathrm{cont}(T^*|_{\mathrm{cols}\le j-1})} + \rho$$

$$\begin{aligned} &= \theta\big(\nu + \mathrm{cont}(T|_{\mathrm{cols}\ge j}) + \rho\big) + \theta\big(\mathrm{cont}(T|_{\mathrm{cols}\le j-1})\big) \\ &= \underbrace{\nu + \mathrm{cont}(T|_{\mathrm{cols}\ge j}) + \rho}_{\mathrm{cont}(T^*|_{\mathrm{cols}\ge j})} + \mathrm{cont}(T^*|_{\mathrm{cols}\le j-1}) \end{aligned}$$

OK

claim proved ✓

By claim

$$a_{\nu+\mathrm{cont}(T)+\rho} = -a_{\nu+\mathrm{cont}(T^*)+\rho}$$

In summary,

T, T^* are col strict Tabeaux shp λ/μ

that fail the conds in thm statement

the choice of j, k is same for T, T^*

the map $T \to T^*$ is an involution

that shows all summands in $*$ cancel except for

those T from thm statement. $\square$

Note: The above thm implies

$$s_\lambda(x_1 \ldots x_n) = \frac{a_{\lambda+\rho}}{a_\rho} \qquad \lambda \in Par, \ \ell(\lambda) \le n$$

Reason: In above thm take

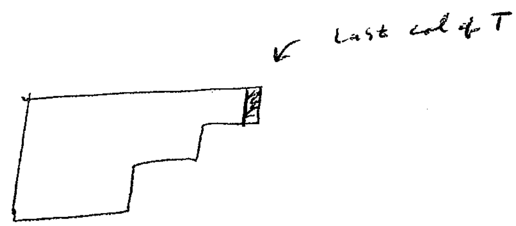
$\mu = \phi, \quad \nu = \phi$

Get

$$a_\rho s_\lambda(x_1 \ldots x_n) = \sum_T a_{cont(T)+\rho}$$

sum over all col strict tableaux T shp λ st $cont(T|col_{\ge j})$ is a partition $\forall j$

Consider a summand T :



content (last col of T) is a partition

So last col has a box containing 1

Must be top box

Top row of T is $11 \cdots 1$

Similarly row i of T is $ii i \cdots i$ $\quad \forall i$

T is unique and $cont(T) = \lambda$

So $a_\rho s_\lambda(x_1 \ldots x_n) = a_{\lambda+\rho}$ ✓

<u>Thm</u> For $\lambda, \mu, \nu \in Par$ (with atmost n pts)

$$s_\nu \, s_{\lambda/\mu} = \sum_T s_{\nu + cont(T)}$$

sum over all col strict tableaux T of shp λ/μ with entries in $\{1, 2, \dots, n\}$ st $\nu + cont(T|col \geq j)$ is a partition $\forall j \geq 1$

pf We have

$$\underbrace{\frac{a_{\nu+\rho}}{a_\rho}}_{= s_\nu} s_{\lambda/\mu} = \sum_T \underbrace{\frac{a_{\nu + cont(T) + \rho}}{a_\rho}}_{= s_{\nu + cont(T)}}$$

$\square$

Cor For $\lambda, \mu \in Par$

$$s_{\lambda/\mu} = \sum_T s_{cont(T)}$$

sum over all col strict tableaux T
with entries in $\{1,\ldots,n\}$ of shp λ/μ
st $cont(T|_{col \geq j})$ is part $\forall j$

pf set $\nu = \phi$ in prev thm. $\square$

We now return to sym functions in ∞ many vars $x_1, x_2, \ldots$

<u>thm</u> For $\lambda, \mu, \nu \in Par$

$$s_\nu \, s_{\lambda/\mu} = \sum_T s_{\nu + \text{cont}(T)}$$

sum over all col strict tableaux T
of shp λ/μ st
$\nu + \text{cont}(T/\text{col} \geq j)$ is a partition $\forall j \geq 1$

pf In the version for vars $x_1, x_2, \ldots x_n$ let $n \to \infty$ $\square$

<u>Cor</u> For $\lambda, \mu \in Par$

$$s_{\lambda/\mu} = \sum_T s_{\text{cont}(T)}$$

sum over all col strict tableaux T
of shp λ/μ st
$\text{cont}(T/\text{cols} \geq j)$ is a partition $\forall j \geq 1$

pf set $\nu = \phi$ in prev thm. $\square$

COR For $\lambda, \mu, \nu \in Par$

$c^{\lambda}_{\mu,\nu}$ = # col strict tableaux T of shp λ/μ and content ν st cont$(T|col_{\geq j})$ is a partition $\forall j \geq 1$

pf By prev Cor

$$s_{\lambda/\mu} = \sum_T s_{cont(T)}$$

recall $\|$

$$\sum_\nu c^{\lambda}_{\mu,\nu} s_\nu$$

$\square$

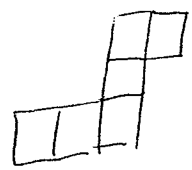
Ex $\lambda = \begin{smallmatrix}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\end{smallmatrix}$

$\mu = \begin{smallmatrix}\bullet&\bullet\\ \bullet&\bullet\end{smallmatrix}$

$\nu = \begin{smallmatrix}\bullet&\bullet&\bullet\\ \bullet&\bullet\\ \bullet\end{smallmatrix}$

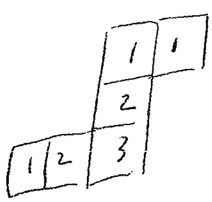
Find $c^{\lambda}_{\mu\nu}$

Col strict tableaux T shp λ/μ

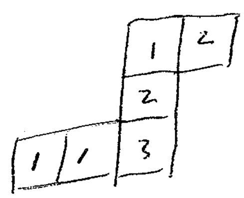


Content ν: Fill with

3 1's
2 2's
1 3



A



B

$cnt(T/col \leq j)$

j	A	B	
1	3 2 1	3 2 1	
2	2 2 1	2 2 1	
3	2 1 1	1 2 1	No
4	1	0 2	No

ok

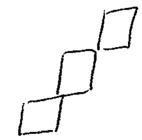
Fails

$c^{\lambda}_{\mu\nu} = 1$

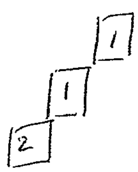
Ex $\quad \lambda = \begin{smallmatrix}\cdot&\cdot&\cdot\\ \cdot&\cdot\\ \cdot\end{smallmatrix} \qquad \mu = \begin{smallmatrix}\cdot&\cdot\\ \cdot\end{smallmatrix} \qquad \nu = \begin{smallmatrix}\cdot&\cdot\\ \cdot\end{smallmatrix}$

Find $c^{\lambda}_{\mu\nu}$

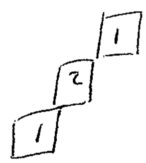
Col strict tableaux T shp λ/μ :



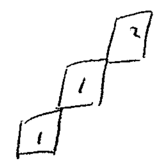
content ν $\qquad$ Fill with $\quad$ 2 1's
$\qquad\qquad\qquad\qquad\qquad\qquad\;\;$ 1 2.



A



B



C

cnt(T|cols ≥ j)

j	A	B	C	
1	21	21	21	
2	2	11	11	
3	1	1	01	NO
	ok	ok	Fails	

$c^{\lambda}_{\mu\nu} = 2$