Lec 26 Friday Nov 4

11/4/16

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Next goal: For $n \in \mathbb{N}$ and $\lambda \in Par$

the Schur function

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \le i,j \le n}}{\det\left(x_i^{n-j}\right)_{1 \le i,j \le n}} \qquad *$$

provided $\ell(\lambda) \le n$

<u>Notation</u> For

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$$

define

$$a_\alpha = \det\left(x_i^{\alpha_j}\right)_{1 \le i,j \le n} \qquad \text{"alternant for } \alpha\text{"}$$

Also define

$$\rho = (n-1, n-2, \ldots, 2, 1, 0) \in \mathbb{N}^n$$

Formula $*$ becomes

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho}$$

provided $\ell(\lambda) \le n$.

Warmup: the case λ has a single part

$\lambda = (a)$

Observe $\quad A_\lambda(x_1, x_2, \ldots, x_n) = h_a(x_1, x_2, \ldots, x_n)$

LEM For $a, n \in \mathbb{N}$

$$h_a(x_1, x_2, \ldots, x_n) = \frac{\begin{vmatrix} x_1^{a+n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{a+n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix}}$$

Pf For notational conv take $n = 3$

$x = x_1 \quad y = x_2 \quad z = x_3$

show

$$h_a(x, y, z) = \frac{\begin{vmatrix} x^{a+2} & x & 1 \\ y^{a+2} & y & 1 \\ z^{a+2} & z & 1 \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}}$$

We will use the gen function

$$\sum_{a \in \mathbb{N}} h_a(x, y, z)\, t^a = \frac{1}{1-xt} \frac{1}{1-yt} \frac{1}{1-zt}$$

Fa an indet t

$$\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = \left| \begin{pmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{pmatrix} \begin{pmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix} \right|$$

$$= \begin{vmatrix} x^2 & x(1-xt) & 1-xt \\ y^2 & y(1-yt) & 1-yt \\ z^2 & z(1-zt) & 1-zt \end{vmatrix}$$

$$= (1-xt)(1-yt)(1-zt) \begin{vmatrix} \frac{x^2}{1-xt} & x & 1 \\ \frac{y^2}{1-yt} & y & 1 \\ \frac{z^2}{1-zt} & z & 1 \end{vmatrix}$$

$$\begin{vmatrix} \frac{x^2}{1-xt} & x & 1 \\ \frac{y^2}{1-yt} & y & 1 \\ \frac{z^2}{1-zt} & z & 1 \end{vmatrix} = \left| \sum_{a \in \mathbb{N}} \begin{pmatrix} x^{a+2} t^a & x & 1 \\ y^{a+2} t^a & y & 1 \\ z^{a+2} t^a & z & 1 \end{pmatrix} \right|$$

$$= \sum_{a \in \mathbb{N}} \begin{vmatrix} x^{a+2} & x & 1 \\ y^{a+2} & y & 1 \\ z^{a+2} & z & 1 \end{vmatrix} t^a$$

So

$$\sum_{a \in \mathbb{N}} h_a(x,y,z) t^a = \frac{1}{1-xt} \frac{1}{1-yt} \frac{1}{1-zt}$$

$$= \sum_{a \in \mathbb{N}} \frac{\begin{vmatrix} x^{a+2} & x & 1 \\ y^{a+2} & y & 1 \\ z^{a+2} & z & 1 \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}} t^a$$

Result follows.

$\square$

Now the general case

Thm For $n \in \mathbb{N}$ and $\lambda \in Par$

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho}$$

provided $\ell(\lambda) \leq n$

pf Consider vars

$x: \; x_1, x_2, \dots, x_n$

$y: \; y_1, y_2, \dots y_n$

Recall

$$\sum_{\lambda \in Par} s_\lambda(x) s_\lambda(y) = \sum_{\mu \in Par} m_\mu(x) h_\mu(y)$$

[suppress y - notation and write $s_\lambda(y) = s_\lambda$, $h_\mu(y) = h_\mu$ etc]

For $\lambda \in Par$

$s_\lambda(x) = 0$ if $\ell(\lambda) > n$

For $\mu \in Par$

$m_\mu(x) = 0$ if $\ell(\mu) > n$

So

$$\sum_{\substack{\lambda \in Par \\ \ell(\lambda) \le n}} s_\lambda(x)\, s_\lambda = \sum_{\substack{\mu \in Par \\ \ell(\mu) \le n}} m_\mu(x)\, h_\mu$$

Mult both sides by a_ρ

Suf to show

$$\sum_{\substack{\lambda \in Par \\ \ell(\lambda) \le n}} a_{\lambda+\rho} \underbrace{s_\lambda}_{\displaystyle \| } \overset{?}{=} a_\rho \sum_{\substack{\mu \in Par \\ \ell(\mu) \le n}} m_\mu(x) \underbrace{h_\mu}_{\displaystyle \|} \qquad (*)$$

$$h_{\mu_1} h_{\mu_2} \cdots h_{\mu_n}$$

$$\left| h_{\lambda_i + j - i} \right|_{1 \le i, j \le n}$$

$$\|$$

$$\sum_{\sigma \in S_n} sgn(\sigma)\, h_{\lambda_1 + \sigma(1) - 1} h_{\lambda_2 + \sigma(2) - 2} \cdots h_{\lambda_n + \sigma(n) - n}$$

For $\mu \in Par$ st $\ell(\mu) \le n$,

compare the h_μ-coef on each side of $(*)$

First assume $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ has all parts distinct

LHS: h_μ - coef is

$$\sum_{\lambda, \sigma} a_{\lambda+\rho} \, \mathrm{sgn}(\sigma)$$

sum over all $\lambda \in Par$ and $\sigma \in S_n$

st $\ell(\lambda) \leq n$ and

$$(\lambda_1 + \sigma(1) - 1, \lambda_2 + \sigma(2) - 2, \ldots, \lambda_n + \sigma(n) - n) \qquad (\star)$$

is a perm of

$$(\mu_1, \mu_2, \ldots, \mu_n)$$

For each summand λ, σ

$$\lambda + \rho = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n + 0)$$

obs

$$\begin{aligned} \star &= (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n + 0) \\ &\quad - (n - \sigma(1), n - \sigma(2), \ldots, n - \sigma(n)) \\ &= \lambda + \rho - \sigma(\rho) \end{aligned}$$

So $\quad \lambda + \rho - \sigma(\rho)$ is perm of μ

Write

$$\gamma(\lambda + \rho - \sigma(\rho)) = \mu \qquad \gamma \in S_n$$

So

$$\gamma(\lambda + \rho) = \mu + \gamma\sigma(\rho)$$

Our h_μ - coeff $= \displaystyle\sum_{\lambda, \sigma} a_{\lambda + \rho} \operatorname{sgn}(\sigma)$

$$= \sum_{\lambda, \sigma} \underbrace{a_{\gamma(\lambda+\rho)}}_{\substack{\| \\ a_{\mu + \gamma\sigma(\rho)}}} \underbrace{\operatorname{sgn}(\gamma)\operatorname{sgn}(\sigma)}_{\substack{\| \\ \operatorname{sgn}(\gamma\sigma)}}$$

[change vars $\varepsilon = \gamma\sigma$]

$$= \sum_{\varepsilon \in S_n} a_{\mu + \varepsilon(\rho)} \operatorname{sgn}(\varepsilon)$$

RHS h_μ-coef is

$a_\rho m_\mu(x)$ $\qquad$ $m_\mu(x)$

multiply

		$x_1^{\mu_1}x_2^{\mu_2}\cdots x_n^{\mu_n}$ + perms
a_ρ	$x_1^{n-1}x_2^{n-2}\cdots x_n^{0}$ + signed perms	

product is

$$\sum_{\varepsilon\in S_n} a_{\mu+\varepsilon(\rho)}\ \mathrm{sgn}(\varepsilon)$$

So for LHS, RHS the h_μ-coef are same.

For the case in which the parts of μ are not all the same, the pf is similar. $\square$

Here is another pf due to MacDonald:

Recall the elem sym functions

$$e_j = e_j(x_1, x_2, \dots, x_n)$$

$$= \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} x_{i_1} x_{i_2} \cdots x_{i_j}$$

For $0 \le i \le n-1$ and $1 \le j \le n$ define

$$e_i^{(j)} = e_i(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots x_n)$$

Define $n \times n$ matrix

$$E = \left((-1)^{n-i} e_{n-i}^{(j)} \right)_{1 \le i, j \le n}$$

For $\mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{N}^n$

define

$$A_\mu = \left(x_j^{\mu_i} \right)_{1 \le i, j \le n} = \begin{pmatrix} x_1^{\mu_1} & \cdots & x_n^{\mu_1} \\ \vdots & & \vdots \\ x_1^{\mu_n} & \cdots & x_n^{\mu_n} \end{pmatrix}$$

$$H_\mu = \left(h_{\mu_i + j - n} \right)_{1 \le i, j \le n} = \begin{pmatrix} h_{\mu_1 + 1 - n} & \cdots & h_{\mu_1} \\ & & h_{\mu_2} \\ \vdots & & \vdots \\ h_{\mu_n + 1 - n} & \cdots & h_{\mu_n} \end{pmatrix}$$

claim

$$A_\mu = H_\mu E$$

pf cl For $1 \le j \le n$ define gen function

$$E^{(j)}(t) = \sum_{i=0}^{n-1} e_i^{(j)} t^i$$

$$= (1+x_1t)(1+x_2t)\cdots(1+x_{j-1}t)(1+x_{j+1}t)\cdots(1+x_nt)$$

Recall

$$\sum_{a=0}^{\infty} h_a t^a = \frac{1}{1-x_1t}\,\frac{1}{1-x_2t}\cdots\frac{1}{1-x_nt}$$

$$\parallel$$

$$H(t)$$

So $$H(t)\,E^{(j)}(-t) = \frac{1}{1-x_jt} = 1 + x_jt + x_j^2t^2 + \cdots$$

In this equation compare coef of t^{μ_i}:

$$h_{\mu_i} e_0^{(j)} - h_{\mu_i - 1} e_1^{(j)} + \cdots + (-1)^{n-1} h_{\mu_i - n + 1} e_{n-1}^{(j)}$$

$$= x_j^{\mu_i}$$

This says the (i,j)-entry of A_μ and $H_\mu E$ $1 \le i,j \le n$

agree for $1 \le i,j \le n$.

claim proved ✓

Now

$$\det A_\mu = \det H_\mu \det E$$

<u>take $\mu = \rho$</u>

$$\det A_\rho = a_\rho$$

$$H_\rho = \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$$

$$\det(H_\rho) = 1$$

So $\quad \det E = a_\rho$

take $\mu = \lambda + \rho$

$$\det A_{\lambda+\rho} = a_{\lambda+\rho}$$

$$H_{\lambda+\rho} = \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+n-1} \\ & h_{\lambda_2} & & h_{\lambda_2+n-2} \\ & & \ddots & \vdots \\ \vdots & & & \\ h_{\lambda_n-n+1} & \cdots & & h_{\lambda_n} \end{pmatrix}$$

$$\det(H_{\lambda+\rho}) = s_\lambda(x_1, \dots, x_n)$$

So

$$a_{\lambda+\rho} = s_\lambda(x_1, \dots, x_n)\, a_\rho$$

So

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho}$$

$\square$