Lec 25 Wed Nov 2 11/2/16

Compare mult, comult

Write
$$s_\mu s_\nu = \sum_{\lambda \in Par} c_{\mu\nu}^{\lambda} s_\lambda$$

$$\Delta(s_\lambda) = \sum_{\mu, \nu \in Par} \gamma_{\mu\nu}^{\lambda} s_\mu \otimes s_\nu$$

Show
$$c_{\mu\nu}^{\lambda} = \gamma_{\mu\nu}^{\lambda} \qquad \lambda, \mu, \nu \in Par$$

To see this, consider the variables

x : $x_1, x_2 \ldots$

y : $y_1, y_2 \ldots$

z : $z_1, z_2, \ldots$

Obs

$$\sum_{\lambda,\mu,\nu\in Par} c^{\lambda}_{\mu,\nu}\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z)$$

$$= \sum_{\mu,\nu\in Par} \underbrace{\left(\sum_{\lambda\in Par} c^{\lambda}_{\mu,\nu}\, s_\lambda(x)\right)}_{\substack{\| \\ s_\mu(x)\, s_\nu(x)}} s_\mu(y)\, s_\nu(z)$$

$$= \left(\sum_{\mu\in Par} s_\mu(x)\, s_\mu(y)\right)\left(\sum_{\nu\in Par} s_\nu(x)\, s_\nu(z)\right)$$

$$= \left(\prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \frac{1}{1-x_i y_j}\right)\left(\prod_{k=1}^{\infty}\prod_{l=1}^{\infty} \frac{1}{1-x_k z_l}\right)$$

[define a sequence

w: $y_1, z_1, y_2, z_2, \ldots$]

$$= \prod_{i=1}^{\infty}\prod_{j=1}^{\infty} \frac{1}{1-x_i w_j}$$

$$= \sum_{\lambda\in Par} s_\lambda(x)\, \underbrace{s_\lambda(w)}_{\substack{\| \\ \sum_{\mu,\nu\in Par} \gamma^{\lambda}_{\mu,\nu}\, s_\mu(y)\, s_\nu(z)}} \quad \text{by def of } \Delta$$

$$= \sum_{\lambda,\mu,\nu\in Par} \gamma^{\lambda}_{\mu,\nu}\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z)$$

Comparing coeffs we find $c^{\lambda}_{\mu,\nu} = \gamma^{\lambda}_{\mu,\nu}$ $\forall\, \lambda, \mu, \nu$ $\square$

<u>Note</u> the $c^{\lambda}_{\mu,\nu}$ are called the <u>Littlewood — Richardson</u> <u>coefs</u>.

We have
$$c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$$

since $s_\mu s_\nu = s_\nu s_\mu$

LEM For $\lambda, \mu \in Par$

$$s_{\lambda/\mu} = \sum_{\nu \in Par} c^{\lambda}_{\mu,\nu} s_\nu$$

↑ zero if $\mu \not\subseteq \lambda$

pf
Recall
$$\Delta(s_\lambda) = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_{\lambda/\mu}$$
$$= \sum_{\mu \in Par} s_\mu \otimes s_{\lambda/\mu}$$

Also
$$\Delta(s_\lambda) = \sum_{\mu,\nu \in Par} c^{\lambda}_{\mu,\nu} s_\mu \otimes s_\nu$$

Result follows. □

LEM Given $\lambda, \mu, \nu \in Par$ st

$$c^{\lambda}_{\mu,\nu} \neq 0.$$

then

(i) $\mu \subseteq \lambda$

(ii) $\nu \subseteq \lambda$

(iii) $|\mu| + |\nu| = |\lambda|$

pf (i) we have

$$\underset{\neq 0}{s_{\lambda/\mu}} = \underbrace{\sum_{\ell \in Par} c^{\lambda}_{\mu\ell} \, s_\ell}_{\neq 0 \text{ since } c^{\lambda}_{\mu,\nu} \text{ makes contrib}}$$

So $\mu \subseteq \lambda$

(ii) By (i) and

$$c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$$

(iii) Since $\Lambda = \bigoplus_{n \in \mathbb{N}} \Lambda_n$ is a Hopf alg grading

$\square$

An alternate approach to Schur functions

Motivation Recall the Kostka numbers K^{λ}_{μ}

For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$

$$s_{\lambda} = \sum_{\mu \in \mathrm{Par}_n} K^{\lambda}_{\mu} \, m_{\mu}$$

ex $n = 3$

	$s_{(1,1,1)}$	$s_{(2,1)}$	$s_{(3)}$
$m_{(1,1,1)}$	1	2	1
$m_{(2,1)}$	0	1	1
$m_{(3)}$	0	0	1

"Kostka matrix"

Fix $n \in \mathbb{N}$. Suppose we wish to study

$$\Lambda_0 + \Lambda_1 + \cdots + \Lambda_n$$

Recall variables

$$x : \quad x_1, x_2, x_3, \ldots$$

View
$$\Lambda = \Lambda(x) \leq R(x)$$

Consider k-alg hom $\qquad *$

$$R(x) \longrightarrow k[x_1, x_2, \ldots, x_n]$$

$$x_i \longrightarrow \begin{cases} x_i & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

$*$ induces k-alg hom $\qquad **$

$$\Lambda \longrightarrow \underbrace{k[x_1, x_2, \ldots, x_n]^{S_n}}_{\text{elements fixed by everything in } S_n}$$

k-module

$$\Lambda = k[x_1, x_2, \ldots, x_n]^{S_n}$$

is graded (by total degree) and finite type

with k-basis

$$m_\lambda \qquad \lambda \in Par \qquad \ell(\lambda) \leq n$$

$\uparrow$
#parts of λ

For $\lambda \in Par$ $**$ sends

$$m_\lambda \longrightarrow \begin{cases} m_\lambda & \text{if } \ell(\lambda) \leq n \\ 0 & \text{if } \ell(\lambda) > n \end{cases}$$

Obs $**$ is surj

obs $\ker(**)$ has k-basis

$$m_\lambda \qquad \lambda \in Par \qquad \ell(\lambda) > n$$

So $\ker(**)$ has 0 intersection with

$$\Lambda_0 + \Lambda_1 + \cdots + \Lambda_n$$

To work with $\Lambda_0 + \Lambda_1 + \cdots + \Lambda_n$, useful to consider

its image under $**$.

Until further notice assume

k = ring $\mathbb{Z}$ or k = a field with char $\neq 2$

Recall the alternating group A_n

A_n is a normal subgp of S_n and $S_n / A_n \cong \mathbb{Z}_2 \overset{\text{view}}{=} \{1, -1\}$

the quot map

$$S_n \to S_n/A_n$$

induces a gp hom

$$sgn : S_n \longrightarrow \mathbb{Z}_2 = \{1, -1\}$$

For $f \in k[x_1, x_2, \ldots, x_n]$

call f <u>alternating</u> whenever

$$\sigma(f) = sgn(\sigma) f \qquad \forall \sigma \in S_n$$

Define

$$\Lambda^{sgn} = \{ f \in k[x_1, \ldots, x_n] \mid f \text{ is alternating} \}$$

Obs

$$K[x_1, \dots, x_n]^{A_n} = \Lambda + \Lambda^{sgn} \qquad \text{(ds of } K\text{-modules)}$$

We have

$$\Lambda \Lambda \subseteq \Lambda$$

$$\Lambda \Lambda^{sgn} \subseteq \Lambda^{sgn}$$

$$\Lambda^{sgn} \Lambda^{sgn} \subseteq \Lambda$$

Recall Vandermonde det

$$\begin{vmatrix} x_1^{n-1} & \cdots & x_1^2 & x_1 & 1 \\ x_2^{n-1} & \cdots & x_2^2 & x_2 & 1 \\ \vdots & & \vdots & \vdots & \vdots \\ x_n^{n-1} & \cdots & x_n^2 & x_n & 1 \end{vmatrix} = \underbrace{\prod_{1 \le i < j \le n} (x_i - x_j)}_{\stackrel{\text{def}}{=} D}$$

$$D \in \Lambda^{sgn}$$

Consider the map $\quad (\star)$

$$\Lambda \longrightarrow \Lambda^{sgn}$$

$$f \mapsto Df$$

Polynomial D is homog, total degree $1+2+\cdots+n-1 = \binom{n}{2}$

So ✯ sends

$$\Lambda_r \to \Lambda^{\text{sgn}}_{r+\binom{n}{2}} \qquad r \in \mathbb{N}$$

Apply ✯ to

$$m_\lambda \qquad \lambda \in Par \qquad \ell(\lambda) \leq n$$

ex $n = 3$ Write $x = x_1$, $y = x_2$ $z = x_3$

$$D = (x-y)(x-z)(y-z)$$

$$= x^2y + y^2z + z^2x - xy^2 - yz^2 - zx^2$$

For $a, b, c \in \mathbb{N}$ define

$$[x^a y^b z^c]^- = x^a y^b z^c + y^a z^b x^c + z^a x^b y^c$$
$$- x^a y^c z^b - y^a z^c x^b - z^a x^c y^b$$
$$= \begin{vmatrix} x^a & x^b & x^c \\ y^a & y^b & y^c \\ z^a & z^b & z^c \end{vmatrix}$$

So $D = [x^2 y^1 z^0]^-$

Find

$$D\, m_{(3)}, \qquad D\, m_{(2,1)}, \qquad D\, m_{(1,1,1)}$$

$$m_{(3)} = [x^3] = x^3 + y^3 + z^3$$

$D\, m_{(3)} =$

	x^3	y^3	z^3
x^2y	x^5y	x^2y^4	x^2yz^3
y^2z		etc	
z^2x			
$-xy^2$			
$-yz^2$			
$-zx^2$			

$$= [x^5y]^- - [x^4y^2]^- + [x^3y^2z]^-$$

We similarly find

$$Dm_{\substack{\cdot\cdot\\ \cdot}} = [x^4y^2]^- - 2[x^3y^2z]^-$$

$$Dm_{\vdots} = [x^3y^2z]^-$$

So

$$\begin{array}{c|ccc} & Dm_{\vdots} & Dm_{\substack{\cdot\cdot\\ \cdot}} & Dm_{\cdots} \\ \hline [x^3y^2z]^- & 1 & -2 & 1 \\ [x^4y^2]^- & 0 & 1 & -1 \\ [x^5y]^- & 0 & 0 & 1 \end{array}$$

take inverse matrix :

	$[x^3y^2z]^-$	$[x^4y^2]^-$	$[x^5y]^-$
$D\,m_{(1,1,1)}$	1	2	1
$D\,m_{(2,1)}$	0	1	1
$D\,m_{(3)}$	0	0	1

Kostka matrix!

So

$$[x^5y]^- = D\left(m_{(1,1,1)} + m_{(2,1)} + m_{(3)}\right)$$
$$= D\, s_{(3)}(x,y,z)$$

$$[x^4y^2]^- = D\left(2m_{(1,1,1)} + m_{(2,1)}\right)$$
$$= D\, s_{(2,1)}(x,y,z)$$

$$[x^3y^2z]^- = D\, m_{(1,1,1)}$$
$$= D\, s_{(1,1,1)}(x,y,z)$$

So

$$s_{\square\square\square}(x,y,z) = \frac{[x^5y]^-}{D} = \frac{\begin{vmatrix} x^5 & x & 1 \\ y^5 & y & 1 \\ z^5 & z & 1 \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}}$$

$$s_{\substack{\square\square\\\square}}(x,y,z) = \frac{[x^4y^2]^-}{D} = \frac{\begin{vmatrix} x^4 & x^2 & 1 \\ y^4 & y^2 & 1 \\ z^4 & z^2 & 1 \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}}$$

$$s_{\substack{\square\\\square\\\square}}(x,y,z) = \frac{[x^3y^2z]^-}{D} = \frac{\begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix}}{\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}}$$

The pattern suggests:

For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}$

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \le i,j \le n}}{\det\left(x_i^{n-j}\right)_{1 \le i,j \le n}}$$

provided $\ell(\lambda) \le n$

We remark

$$s_\lambda(x_1, x_2, \ldots, x_n) = 0 \quad \text{if } \ell(\lambda) > n.$$