Lec 24 Monday Oct 31

10/31/16

1

LEM Assume $\mathbb{Q}$ is subring of K

For $n \in \mathbb{N}$

$$\sum_{\lambda \in Par_n} e_\lambda \otimes m_\lambda = \sum_{\lambda \in Par_n} \frac{p_\lambda \otimes p_\lambda}{z_\lambda} (-1)^{n-\ell(\lambda)} = \sum_{\lambda \in Par_n} m_\lambda \otimes e_\lambda$$

pf

$$\sum_{\lambda} \frac{p_\lambda \otimes p_\lambda}{z_\lambda} (-1)^{n-\ell(\lambda)} \overset{?}{=} \sum_{\lambda} m_\lambda \otimes e_\lambda$$

$$\| \qquad\qquad \|$$

$$\sum_{\lambda,\mu} \frac{m_\mu \otimes p_\lambda}{z_\lambda} (-1)^{n-\ell(\lambda)} b_\mu^\lambda \qquad \sum_{\lambda,\mu} m_\lambda \otimes p_\mu \frac{b_\lambda^\mu}{z_\mu} (-1)^{n-\ell(\mu)}$$

$$\| \quad [\lambda \leftrightarrow \mu]$$

$$\checkmark \qquad \sum_{\lambda,\mu} m_\mu \otimes p_\lambda \frac{b_\mu^\lambda}{z_\lambda} (-1)^{n-\ell(\lambda)}$$

$\square$

thm We have

$$\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}(1+x_i y_j) = \sum_{\lambda\in Par} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}(-1)^{|\lambda|-\ell(\lambda)}$$

pf

$$\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}(1+x_i y_j) = \sum_{\lambda\in Par} e_\lambda(x)\, m_\lambda(y)$$

$$= \sum_{n\in\mathbb{N}}\left(\sum_{\lambda\in Par_n} e_\lambda(x)\, m_\lambda(y)\right)$$

$$= \sum_{n\in\mathbb{N}}\left(\sum_{\lambda\in Par_n} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}(-1)^{n-\ell(\lambda)}\right)$$

$$= \sum_{\lambda\in Par} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}(-1)^{|\lambda|-\ell(\lambda)}$$

$\square$

Assume $\mathbb{Q}$ is a subring of K

For $n \in \mathbb{N}$ consider the following K-bases for Λ_n:

$$s_\lambda \qquad \lambda \in \mathrm{Par}_n \qquad *$$

$$\frac{p_\lambda}{z_\lambda} \qquad \lambda \in \mathrm{Par}_n \qquad **$$

Write $*$ in terms of $**$

$$s_\lambda = \sum_{\mu \in \mathrm{Par}_n} c^\lambda_\mu \frac{p_\mu}{z_\mu}$$

We will see, the transition matrix $(c^\lambda_\mu)_{\lambda,\mu \in \mathrm{Par}_n}$ is the character table of the sym group S_n.

Ex $n=5$

Write $\{s_\lambda\}_\lambda$ in terms of $\{p_\lambda/z_\lambda\}_\lambda$

	$s_{(5)}$	$s_{(4,1)}$	$s_{(3,2)}$	$s_{(3,1,1)}$	$s_{(2,2,1)}$	$s_{(2,1,1,1)}$	$s_{(1,1,1,1,1)}$
$\frac{1}{120} p_{(1,1,1,1,1)}$	1	4	5	6	5	4	1
$\frac{1}{12} p_{(2,1,1,1)}$	1	2	1	0	-1	-2	-1
$\frac{1}{8} p_{(2,2,1)}$	1	0	1	-2	1	0	1
$\frac{1}{6} p_{(3,1,1)}$	1	1	-1	0	-1	1	1
$\frac{1}{6} p_{(3,2)}$	1	-1	1	0	-1	1	-1
$\frac{1}{4} p_{(4,1)}$	1	0	-1	0	1	0	-1
$\frac{1}{5} p_{(5)}$	1	-1	0	1	0	-1	1

(character table for S_5)

A bilinear form $(\,,\,)$ on Λ.

Def. We endow K-module Λ with a K-bilinear form $(\,,\,)$ wrt which the s_λ are orthonormal:

$$(\,,\,) \quad \begin{array}{ccc} \Lambda \times \Lambda & \longrightarrow & K \\ s_\lambda \quad s_\mu & \longrightarrow & \delta_{\lambda,\mu} \end{array}$$

We obs $(\,,\,)$ is symmetric.

By constr $(\Lambda_n, \Lambda_m) = 0$ if $n \neq m$ $n, m \in \mathbb{N}$

Call $(\,,\,)$ the **Hall inner product**

LEM For $u, v \in \Lambda$,

(i) $(\omega(u), v) = (u, \omega(v))$

$\omega =$ fund invol

(ii) $(S(u), v) = (u, S(v))$

pf WLOG $u = s_\lambda$, $v = s_\mu$ $\lambda, \mu \in Par$

(i) $$\underbrace{(\underset{\overset{\|}{s_{\lambda^t}}}{\omega(s_\lambda)}, s_\mu)}_{\overset{\|}{\delta_{\lambda^t, \mu}}} \overset{?}{=} \underbrace{(s_\lambda, \underset{\overset{\|}{s_\mu}}{\omega(s_\mu)})}_{\overset{\|}{\lambda_{\lambda, \mu^t}}}$$

$$\delta_{\lambda^t,\mu} \underset{\text{ok}}{=} \lambda_{\lambda,\mu^t}$$

(ii) Sim $\square$

LEM We have

$$(m_\lambda, h_\mu) = \delta_{\lambda,\mu} \qquad \lambda, \mu \in Par$$

pf WLOG $|\lambda| = |\mu|$ $(= n)$

Consider these k-bases for Λ_n:

$\{s_\lambda\}_\lambda$ $\qquad$ $\{h_\lambda\}_\lambda$

$\{m_\lambda\}_\lambda$ $\qquad$ $\{m_\lambda^*\}_\lambda$

$\uparrow$ dual basis wrt $(,)$

Show $h_\lambda = m_\lambda^*$ $\forall \lambda$

Consider these transition matrices:

$$\{m_\lambda\}_\lambda \to \{s_\lambda\}_\lambda \qquad (1)$$

$$\{s_\lambda\}_\lambda \to \{h_\lambda\}_\lambda \qquad (2)$$

$$\{s_\lambda\}_\lambda \to \{m_\lambda^*\}_\lambda \qquad (3)$$

Recall

$$s_\lambda = \sum_\mu K^\lambda_\mu m_\mu \qquad \forall \lambda$$

$$h_\lambda = \sum_\mu K^\mu_\lambda s_\mu$$

So (1), (2) are transpose.

By lin alg and since $s_\lambda^* = s_\lambda$

(1), (3) are transpose.

So (2) = (3) and result follows. $\square$

LEM We have

$$(w_\lambda, e_\mu) = \delta_{\lambda,\mu} \qquad \lambda, \mu \in Par$$

pf obs

$$\begin{aligned}(w_\lambda, e_\mu) &= (\omega(m_\lambda), \omega(h_\mu)) \\ &= (\omega^2(m_\lambda), h_\mu) \\ &= (m_\lambda, h_\mu) \\ &= \delta_{\lambda,\mu} \qquad \square\end{aligned}$$

LEM Assume $\mathbb{Q}$ is a subring of K

then

$$(p_\lambda, p_\mu) = \delta_{\lambda,\mu} z_\lambda \qquad \forall \lambda, \mu \in \mathrm{Par}$$

pf WLOG $|\lambda| = |\mu| (= n)$

Consider these K-bases for Λ_n

$$\{m_\lambda\}_\lambda, \qquad \{h_\lambda\}_\lambda$$

$$\{p_\lambda\}_\lambda \qquad \{p_\lambda^*\}_\lambda, \qquad \left\{\frac{p_\lambda}{z_\lambda}\right\}_\lambda$$

show $p_\lambda^* = \dfrac{p_\lambda}{z_\lambda} \quad \forall \lambda$

Consider these trans matrices

$$\{m_\lambda\}_\lambda \longrightarrow \{p_\lambda\}_\lambda \qquad (1)$$

$$\left\{\frac{p_\lambda}{z_\lambda}\right\}_\lambda \longrightarrow \{h_\lambda\}_\lambda \qquad (2)$$

$$\{p_\lambda^*\}_\lambda \longrightarrow \{h_\lambda\}_\lambda \qquad (3)$$

Recall

$$p_\lambda = \sum_\mu b^\lambda_\mu \, m_\mu$$

$$h_\lambda = \sum_\mu \frac{b^\mu_\lambda}{z_\mu} \, p_\mu$$

So (1), (2) are transpose.

By lin alg and since $m^*_\lambda = h_\lambda$

(1), (3) are transpose.

So (2) = (3) and result follows. $\square$

COR Assume the field $\mathbb{R}$ is a subring of K

then the elements

$$\frac{p_\lambda}{z_\lambda^{1/2}} \qquad \lambda \in \mathrm{Par}$$

form a K-basis for Λ that is orthonormal wrt $(\,,\,)$ $\square$

Recall the restricted dual Hopf alg

$$\Lambda^0 = \bigoplus_{n \in \mathbb{N}} \Lambda_n^*$$

Fn $n \in \mathbb{N}$ $\exists$ K-module iso

$$\theta_n : \Lambda_n \to \Lambda_n^*$$

st fn $x, y \in \Lambda_n$

$$(x, y) = \theta_n(x)(y)$$

$\exists$ K module iso

$$\theta : \Lambda \to \Lambda^0$$

st $\theta|_{\Lambda_n} = \theta_n$ $\forall n \in \mathbb{N}$

Next goal: show θ is a Hopf algebra iso.

<u>Aside</u> Given a Hopf algebra H over K

that is finite-free

Given a K-basis for H: $\{v_i\}_i$

dual K-basis for H^*: $\{v_i^*\}_i$

Compare Hopf algebras H, H^*:

	H	H^*
mult	$v_i v_j = \sum_h c_{ij}^h v_h$	$v_i^* \star v_j^* = \sum_h \gamma_{ij}^h v_h^*$
comult	$\Delta(v_h) = \sum_{i,j} \gamma_{ij}^h v_i \otimes v_j$	$\Delta(v_h^*) = \sum_{i,j} c_{ij}^h v_i^* \otimes v_j^*$
unit	$1 = \sum_i \alpha_i v_i$	$\sum_i \beta_i v_i^*$
counit	$\varepsilon = \sum_i \beta_i v_i^*$	$\sum_i \alpha_i v_i$ (view $(H^*)^* = H$)
antipode	$S(v_j) = \sum_i \sigma_{ij} v_i$	$S^*(v_j^*) = \sum_i \sigma_{ji} v_i^*$

pf check assertions for H^*

mult

$$v_i^* \star v_j^* \overset{?}{=} \sum_h \gamma_{ij}^h \, v_h^*$$

$$\forall r \quad (v_i^* \star v_j^*)(v_r) \overset{?}{=} \underbrace{\sum_h \gamma_{ij}^h \underbrace{v_h^*(v_r)}_{\delta_{hr}}}_{\gamma_{ij}^r}$$

$$\|$$

$$\left[\Delta(v_r) = \sum_{s,t} \gamma_{st}^r \, v_s \otimes v_t\right]$$

$$\underbrace{\sum_{s,t} \gamma_{st}^r \underbrace{v_i^*(v_s)}_{\delta_{is}} \underbrace{v_j^*(v_t)}_{\delta_{jt}}}_{\gamma_{ij}^r}$$

ok

<u>Comult</u>

$$\Delta(v_h^*) \overset{?}{=} \sum_{i,j} c_{ij}^h \, v_i^* \otimes v_j^*$$

Recall def of Δ

$$\Delta : \quad H^* \xrightarrow{m^*} (H \otimes H)^* \xrightarrow{\text{can}} H^* \otimes H^*$$

$$\phi \hookleftarrow f \otimes g$$

$$\phi(a \otimes b) = f(a)\, g(b)$$

$$v_h^* \longrightarrow m^*(v_h^*) \underset{?}{\xrightarrow{\text{can}}} \sum_{i,j} c_{ij}^h \, v_i^* \otimes v_j^*$$

$\forall s, t$

$$m^*(v_h^*)\,(v_s \otimes v_t) \overset{?}{=} \underbrace{\sum_{i,j} c_{ij}^h \, \underset{\delta_{is}}{\underset{\parallel}{v_i^*(v_s)}} \, \underset{\delta_{jt}}{\underset{\parallel}{v_j^*(v_t)}}}_{\parallel \atop c_{st}^h}$$

$$\parallel$$

$$v_h^*\big(m(v_s \otimes v_t)\big)$$

$$\parallel$$

$$v_h^*(v_s v_t)$$

$$\parallel$$

$$\underbrace{v_h^*\Big(\sum_r c_{st}^r v_r\Big)}_{\parallel \atop c_{st}^h}$$

OK

The assertions about H^* for unit, counit, antipode

we saw earlier. $\square$

With above notation endow H with a

K-bilinear form $(\, , \,)$ st

$$(v_i, v_j) = \delta_{ij} \qquad \forall i, j$$

$\exists$ H-module iso

$$\theta : H \to H^*$$

st $\forall$ $x, y \in H$

$$\theta(x)(y) = (x, y)$$

obs $\theta(v_i) = v_i^* \qquad \forall i$

COR With above notation TFAE

(i) $\theta: H \to H^*$ is a Hopf alg iso

(ii) $c_{ij}^h = \gamma_{ij}^h$

$\alpha_i = \beta_i \quad \forall h, i, j$

$\sigma_{ij} = \sigma_{ji}$

pf Clear $\square$

Above is about the case of H finite free

the case of H graded of finite type is similar (ex).

Back to $\theta : \Lambda \to \Lambda^o$

Thm The map $\theta : \Lambda \to \Lambda^o$ is a Hopf alg iso.

pf Recall Λ is graded of finite type.
Recall the k-bilinear form

$$(s_\lambda, s_\mu) = \delta_{\lambda,\mu} \qquad \lambda, \mu \in Par$$

Apply the version of the prev COR that applies here.

Compare $1, \varepsilon$

$$1 = s_\phi$$

Also

$$\varepsilon(1) = 1,$$

$$\varepsilon(s_\lambda) = 0 \text{ if } \lambda \neq \phi \qquad \lambda \in Par$$

So $\varepsilon = 1^*$ ✓

Antipode S Recall

$$S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda^t} \qquad \lambda \in Par$$

So wrt the k-basis $\{s_\lambda\}_\lambda$ the matrix rep S is symmetric ✓