Lec 23 Fri Oct 28

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LEM For $n \in \mathbb{N}$

$n!\,h_n$ is a nonnegative integral

linear combin of

$p_\lambda \qquad \lambda \in \mathrm{Par}_n$

pf Use induction on n and (**) above

$\square$

Until further notice, assume

$\mathbb{Q}$ is subring of K

Assume $\mathbb{Q}$ is a subring of K

For $n \geq 1$ write h_n in terms of the p_λ:

Using $(**)$,

$$h_1 = p_1$$

$$h_2 = \frac{1}{2} p_2 + \frac{1}{2} p_1^2$$

$$h_3 = \frac{1}{3} p_3 + \frac{1}{2} p_1 p_2 + \frac{1}{6} p_1^3$$

$$h_4 = \frac{1}{4} p_4 + \frac{1}{3} p_1 p_3 + \frac{1}{8} p_2^2 + \frac{1}{4} p_1^2 p_2 + \frac{1}{24} p_1^4$$

$$h_5 = \frac{1}{5} p_5 + \frac{1}{4} p_4 p_1 + \frac{1}{6} p_3 p_2 + \frac{1}{6} p_3 p_1^2 + \frac{1}{8} p_2^2 p_1 + \frac{1}{12} p_2 p_1^3 + \frac{1}{120} p_1^5$$

$$\cdots$$

For $n \geq 1$

$$h_n = \frac{1}{n} p_n + LT$$

(show this by ind on n)

COR For $n \geq 1$ and $\lambda \in \mathrm{Par}_n$

$$h_\lambda - \frac{p_\lambda}{\lambda_1 \lambda_2 \cdots \lambda_\ell} \in \sum_{\mu \triangleleft \lambda} K p_\mu$$

where $\ell = \mathrm{length}(\lambda)$

pf Since

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$$

$\square$

Continue to assume $\mathbb{Q}$ is subring of K

Until further notice, fix $n \in \mathbb{N}$ and write

$$V = \Lambda_n$$

For $\lambda \in \mathrm{Par}\,n$ define

$$_{\trianglelefteq\lambda}V = \sum_{\substack{\mu \in \mathrm{Par}\,n \\ \mu \trianglelefteq \lambda}} K\,p_\mu$$

$$_{\trianglerighteq\lambda}V = \sum_{\substack{\mu \in \mathrm{Par}\,n \\ \mu \trianglerighteq \lambda}} K\,p_\mu$$

Obs each of $_{\trianglelefteq\lambda}V$, $_{\trianglerighteq\lambda}V$ is a finite-free K-module and

$$_{\trianglelefteq\lambda}V \cap {}_{\trianglerighteq\lambda}V = K\,p_\lambda$$

LEM Assume $\mathbb{Q}$ is a subring of K.
For $\lambda \in \mathrm{Par}_n$ each of the following (1)–(3)
is a K-basis for $\trianglelefteq_\lambda V$:

$$p_\mu \qquad \mu \trianglelefteq \lambda \tag{1}$$

$$h_\mu \qquad \mu \trianglelefteq \lambda \tag{2}$$

$$e_\mu \qquad \mu \trianglelefteq \lambda \tag{3}$$

pf (1) By def

(2) By prev Cor

(3) Apply ω to (2) $\square$

LEM. Assume $\mathbb{Q}$ is a subring of $\mathbb{K}$

For $\lambda \in \text{Par}_n$ each of the following (4)–(6) is a $\mathbb{K}$-basis for ${}_{\trianglerighteq\lambda}V$:

$$p_\mu \qquad \mu \trianglerighteq \lambda \tag{4}$$

$$m_\mu \qquad \mu \trianglerighteq \lambda \tag{5}$$

$$\omega_\mu \qquad \mu \trianglerighteq \lambda \tag{6}$$

pf (4) By constr.

(5) Recall

$$p_\lambda = \sum_{\mu \trianglerighteq \lambda} b^\lambda_\mu m_\mu \qquad \text{of } b^\lambda_\lambda \in \mathbb{Q}$$

(6) Apply ω to (5) $\square$

Thm Assume $\mathbb{Q}$ is a subring of K

Fix $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$

(i) the intersection

$$\left(\sum_{\mu \trianglerighteq \lambda} K m_\mu \right) \cap \left(\sum_{\nu \trianglelefteq \lambda} K h_\nu \right) \quad (*)$$

is a finite-free K-module with K-basis p_λ

(ii) p_λ is the unique element of $(*)$ with m_λ-coef b^λ_λ

(iii) p_λ is the unique element of $(*)$ with h_λ-coef $\lambda_1 \lambda_2 \cdots \lambda_\ell$ ($\ell = \text{length } \lambda$)

pf (i) Obs

$$(*) = \trianglerighteq_\lambda V \cap \trianglelefteq_\lambda V = K p_\lambda$$

(ii) Recall

$$p_\lambda = \sum_{\mu \trianglerighteq \lambda} b^\lambda_\mu m_\mu$$

(iii) By prev Cor. $\square$

Next goal: Assume $\mathbb{Q}$ is subring of K.

For variables

x : $x_1, x_2, \ldots$

y : $y_1, y_2, \ldots$

show

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in Par} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}$$

where

$$z_\lambda = m_1!\, 1^{m_1}\, m_2!\, 2^{m_2}\, m_3!\, 2^{m_3} \cdots$$

where m_i = # of parts of λ equal to i

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Assume $\mathbb{Q}$ is subring of $\mathbb{K}$

Fix $n \in \mathbb{N}$ and write $V = \Lambda_n$

Recall $\quad {}_{\trianglelefteq\lambda} V \qquad \lambda \in \mathrm{Par}_n$

${}_{\trianglerighteq\lambda} V$

Define

$$^{+}V = \sum_{\lambda \in \mathrm{Par}_n} {}_{\trianglerighteq\lambda} V \otimes {}_{\trianglelefteq\lambda} V$$

$$^{-}V = \sum_{\lambda \in \mathrm{Par}_n} {}_{\trianglelefteq\lambda} V \otimes {}_{\trianglerighteq\lambda} V$$

LEM We have

(i) ^{+}V is a finite-free $\mathbb{K}$-module with $\mathbb{K}$-basis

$$p_\mu \otimes p_\nu \qquad \mu, \nu \in \mathrm{Par}_n, \quad \mu \trianglerighteq \nu$$

(ii) ^{-}V is a finite-free $\mathbb{K}$-module with $\mathbb{K}$-basis

$$p_\mu \otimes p_\nu \qquad \mu, \nu \in \mathrm{Par}_n, \quad \mu \trianglelefteq \nu$$

(iii) $^{+}V \cap {}^{-}V$ is a finite-free $\mathbb{K}$-module with $\mathbb{K}$-basis

$$p_\mu \otimes p_\mu \qquad \mu \in \mathrm{Par}_n$$

pf Similar to the pf of the corresp result for the V^{+}, V^{-} $\square$

LEM Assume $\mathbb{Q}$ is a subring of K. Fix $n \in \mathbb{N}$

$$\sum_{\lambda \in Par_n} h_\lambda \otimes m_\lambda = \sum_{\lambda \in Par_n} \frac{p_\lambda \otimes p_\lambda}{z_\lambda} = \sum_{\lambda \in Par_n} m_\lambda \otimes h_\lambda$$

pf We have

$$\underbrace{\sum_{\lambda \in Par_n} \overset{\in \trianglelefteq_\lambda V}{h_\lambda} \otimes \overset{\in \trianglerighteq_\lambda V}{m_\lambda}}_{\in\ {}^{-}V} = \underbrace{\sum_{\lambda \in Par_n} \overset{\in \trianglelefteq_\lambda V}{m_\lambda} \otimes \overset{\in \trianglerighteq_\lambda V}{h_\lambda}}_{\in\ {}^{+}V} \quad (= y)$$

So

$$y \in {}^{+}V \cap {}^{-}V$$

$$= \sum_{\lambda \in Par_n} K\, p_\lambda \otimes p_\lambda$$

Write

$$y = \sum_{\lambda \in Par_n} \alpha_\lambda\, p_\lambda \otimes p_\lambda$$

We now compute the α_λ.

We have

$$\sum_{\lambda \in Par_n} \alpha_\lambda \, p_\lambda \otimes p_\lambda = \sum_{\lambda \in Par_n} h_\lambda \otimes m_\lambda$$

$$= \sum_{\lambda \in Par_n} \left(\frac{1}{\lambda_1 \lambda_2 \cdots} p_\lambda + LT \right) \otimes \left(\frac{1}{b_\lambda^\lambda} p_\lambda + LT \right)$$

$$= \sum_{\lambda \in Par_n} \frac{p_\lambda \otimes p_\lambda}{\lambda_1 \lambda_2 \cdots \; b_\lambda^\lambda} + LT.$$

So for $\lambda \in Par_n$.

$$\alpha_\lambda = \frac{1}{\lambda_1 \lambda_2 \cdots \; b_\lambda^\lambda}$$

recall $b_\lambda^\lambda = m_1! \, m_2! \cdots$

obs $\lambda_1 \lambda_2 \cdots = 1^{m_1} \, 2^{m_2} \cdots$

So $\lambda_1 \lambda_2 \cdots \; b_\lambda^\lambda = z_\lambda$.

So $\alpha_\lambda = \dfrac{1}{z_\lambda}$

We conclude

$$\eta = \sum_{\lambda \in Par_n} \frac{p_\lambda \otimes p_\lambda}{z_\lambda}$$

$\square$

thm Assume $\mathbb{Q}$ is a subring of K.

then

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in Par} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}$$

pf obs

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in Par} h_\lambda(x)\, m_\lambda(y)$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\lambda \in Par_n} h_\lambda(x)\, m_\lambda(y) \right)$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\lambda \in Par_n} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda} \right)$$

$$= \sum_{\lambda \in Par} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda}$$

□

LEM Assume $\mathbb{Q}$ is a subring of K.

Fix $n \in \mathbb{N}$ and $\lambda \in Par_n$

$$h_\lambda = \sum_{\mu \in Par_n} p_\mu \frac{b^\mu_\lambda}{z_\mu}$$

pf Recall

$$p_\lambda = \sum_\mu b^\lambda_\mu m_\mu \qquad \lambda \in Par_n$$

write

$$h_\lambda = \sum_\mu \gamma^\lambda_\mu p_\mu$$

show $\gamma^\lambda_\mu = \dfrac{b^\mu_\lambda}{z_\mu}$

$$\sum_{\lambda \in Par_n} h_\lambda \otimes m_\lambda = \sum_{\lambda \in Par_n} \frac{p_\lambda \otimes p_\lambda}{z_\lambda}$$

$$\sum_{\lambda,\mu} p_\mu \otimes m_\lambda \gamma^\lambda_\mu \qquad \sum_{\lambda,\mu} p_\lambda \otimes m_\mu \frac{b^\lambda_\mu}{z_\lambda}$$

$[\lambda \leftrightarrow \mu]$

$$\sum_{\lambda,\mu} p_\mu \otimes m_\lambda \frac{b^\mu_\lambda}{z_\mu}$$

So $\gamma^\lambda_\mu = \dfrac{b^\mu_\lambda}{z_\mu}$ ✓

$\square$

LEM Assume $\mathbb{Q}$ is a subring of K.

For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$

$$e_\lambda = \sum_{\mu \in \mathrm{Par}_n} p_\mu \frac{b_\lambda^\mu}{z_\mu} (-1)^{|\lambda| - \ell(\mu)}$$

pf In the eqn

$$h_\lambda = \sum_{\mu \in \mathrm{Par}_n} p_\mu \frac{b_\lambda^\mu}{z_\mu}$$

apply S to each side and recall

$$S(h_\lambda) = (-1)^{|\lambda|} e_\lambda$$

$$S(p_\mu) = (-1)^{\ell(\mu)} p_\mu$$

$\square$