LEM For $n \in \mathbb{N}$

$$\sum_{\lambda \in Par_n} h_\lambda \otimes m_\lambda = \sum_{\lambda \in Par_n} s_\lambda \otimes s_\lambda = \sum_{\lambda \in Par_n} m_\lambda \otimes h_\lambda$$

pf We have

$$\underbrace{\sum_{\lambda \in Par_n} \overset{V_{\unrhd\lambda}}{\overset{\cup}{h_\lambda}} \otimes \overset{V_{\unlhd\lambda}}{\overset{\cup}{m_\lambda}}}_{\underset{V^+}{\cap}} = \underbrace{\sum_{\lambda \in Par_n} \overset{V_{\unlhd\lambda}}{\overset{\cup}{m_\lambda}} \otimes \overset{V_{\unrhd\lambda}}{\overset{\cup}{h_\lambda}}}_{\underset{V^-}{\cap}} \quad (= z)$$

So $z \in V^+ \cap V^-$

$$= \sum_{\lambda \in Par_n} K\, s_\lambda \otimes s_\lambda$$

Write

$$z = \sum_{\lambda \in Par_n} a_\lambda\, s_\lambda \otimes s_\lambda \qquad a_\lambda \in K$$

In the K-basis

$$s_\mu \qquad \mu \in Par_n$$

the s_λ-coef of each of h_λ, m_λ is 1 for $\lambda \in Par_n$.

therefore $a_\lambda = 1 \qquad \lambda \in Par_n$

So $z = \sum_{\lambda \in Par_n} s_\lambda \otimes s_\lambda$ $\square$

thm We have

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-x_i y_j} = \sum_{\lambda \in Par} s_\lambda(x) s_\lambda(y)$$

pf obs

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-x_i y_j} = \sum_{\lambda \in Par} h_\lambda(x) m_\lambda(y)$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\lambda \in Par_n} h_\lambda(x) m_\lambda(y) \right)$$

by prev LEM

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\lambda \in Par_n} s_\lambda(x) s_\lambda(y) \right)$$

$$= \sum_{\lambda \in Par} s_\lambda(x) s_\lambda(y)$$

$\square$

Recall that for $n \in \mathbb{N}$ and $\lambda \in Par_n$

$$s_\lambda = \sum_{\mu \in Par_n} K^\lambda_\mu \, m_\mu$$

LEM For $n \in \mathbb{N}$ and $\lambda \in Par_n$

$$h_\lambda = \sum_{\mu \in Par_n} K^\mu_\lambda \, s_\mu$$

pf Write

$$h_\lambda = \sum_{\mu \in Par_n} \varepsilon^\lambda_\mu \, s_\mu \qquad \varepsilon^\lambda_\mu \in K$$

show $\varepsilon^\lambda_\mu = K^\mu_\lambda \quad \forall \lambda, \mu$

We have

$$\sum_{\lambda \in Par_n} h_\lambda \otimes m_\lambda = \sum_{\lambda \in Par_n} s_\lambda \otimes s_\lambda$$

$$\sum_{\lambda \in Par_n} h_\lambda \otimes m_\lambda = \sum_{\lambda, \mu} s_\mu \otimes m_\lambda \, \varepsilon^\lambda_\mu$$

$$\sum_{\lambda \in Par_n} s_\lambda \otimes s_\lambda = \sum_{\lambda, \mu} s_\lambda \otimes m_\mu \, K^\lambda_\mu \overset{\lambda \leftrightarrow \mu}{=} \sum_{\lambda, \mu} s_\mu \otimes m_\lambda \, K^\mu_\lambda$$

So $\varepsilon^\lambda_\mu = K^\mu_\lambda \quad \forall \lambda, \mu$

$\square$

LEM For $n \in \mathbb{N}$ and $\lambda \in Par_n$

$$e_\lambda = \sum_{\mu \in Par_n} K_\lambda^{\mu^t} s_\mu$$

pf In the equation

$$h_\lambda = \sum_{\mu \in Par_n} K_\lambda^{\mu} s_\mu$$

apply ω to each side to get

$$e_\lambda = \sum_{\mu \in Par_n} K_\lambda^{\mu} s_{\mu^t}$$

$$[\mu \to \mu^t]$$

$$= \sum_{\mu \in Par_n} K_\lambda^{\mu^t} s_\mu$$

$\square$

For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}\, n$ recall

$$a^{\lambda}_{\mu} = \#\,\big(0,1\big)\text{-matrices with row/col sums}$$

$$\begin{array}{c|c}
 & \mu_1 \ \mu_2 \ \cdots \\
\hline
\lambda_1 & \\
\lambda_2 & \\
\vdots &
\end{array}$$

LEM With above notation

$$a^{\lambda}_{\mu} = \sum_{\nu \in \mathrm{Par}\, n} K^{\nu}_{\lambda} \, K^{\nu^t}_{\mu}$$

Kostka numbers

pf Recall

$$e_\lambda = \sum_{\mu} a^{\lambda}_{\mu} \, m_\mu \qquad (*)$$

Recall also

$$e_\lambda = \sum_{\nu} K^{\nu^t}_{\lambda} \, s_\nu$$

$[\nu \to \nu^t]$

$$= \sum_{\nu} K^{\nu}_{\lambda} \, \underbrace{s_{\nu^t}}_{= \sum_{\mu} K^{\nu^t}_{\mu} m_\mu}$$

$$= \sum_{\mu} m_\mu \Big(\sum_{\nu} K^{\nu}_{\lambda} K^{\nu^t}_{\mu} \Big) \qquad (**)$$

Compare $(*)$, $(**)$ to get result. $\square$

Next goal: For variables

x: $x_1, x_2, \ldots$

y: $y_1, y_2, \ldots$

show

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda \in Par} s_{\lambda^t}(x)\, s_\lambda(y)$$

LEM We have

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda \in Par} e_\lambda(x)\, m_\lambda(y) \qquad (*)$$

$$= \sum_{\lambda \in Par} m_\lambda(x)\, e_\lambda(y) \qquad (**)$$

pf (*) Recall

$$\prod_{i=1}^{\infty} (1+x_i t) = \sum_{n \in \mathbb{N}} e_n t^n$$

So

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1+x_i y_j) = \prod_{j=1}^{\infty} \left(\sum_{n \in \mathbb{N}} e_n y_j^n \right)$$

For $\lambda \in Par$, the coef of $e_\lambda(x)$ above

is $[y_1^{\lambda_1} y_2^{\lambda_2} \cdots]$

which is $m_\lambda(y)$

Result follows. $\square$

(**) Sim.

LEM For $n \in \mathbb{N}$

$$\sum_{\lambda \in Par\,n} e_\lambda \otimes m_\lambda = \sum_{\lambda \in Par\,n} m_\lambda \otimes e_\lambda$$

pf By prev lem

$$\sum_{\lambda \in Par\,n} e_\lambda(x)\, m_\lambda(y) = \sum_{\lambda \in Par\,n} m_\lambda(x)\, e_\lambda(y)$$

$\square$

Result follows.

LEM Fix $n \in \mathbb{N}$

$$\sum_{\lambda \in Par_n} e_\lambda \otimes m_\lambda = \sum_{\lambda \in Par_n} s_{\lambda^t} \otimes s_\lambda = \sum_{\lambda \in Par_n} m_\lambda \otimes e_\lambda$$

$\uparrow$

pf

$$\sum_\lambda s_{\lambda^t} \otimes s_\lambda \overset{?}{=} \sum_\lambda m_\lambda \otimes e_\lambda$$

$$\sum_\lambda s_{\lambda^t} \otimes s_\lambda = \sum_{\lambda,\mu} K_\mu^{\lambda^t} m_\mu \otimes s_\lambda$$

$$\sum_\lambda m_\lambda \otimes e_\lambda = \sum_{\lambda,\mu} m_\lambda \otimes s_\mu \, K_\lambda^{\mu^t}$$

$[\lambda \leftrightarrow \mu]$

$$= \sum_{\lambda,\mu} m_\mu \otimes s_\lambda \, K_\mu^{\lambda^t}$$

$$\sum_{\lambda,\mu} K_\mu^{\lambda^t} m_\mu \otimes s_\lambda \overset{=}{\text{ok}} \sum_{\lambda,\mu} m_\mu \otimes s_\lambda \, K_\mu^{\lambda^t}$$

$\square$

Thm We have

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda \in Par} s_{\lambda^t}(x) \, s_\lambda(y).$$

pf obs

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 + x_i y_j) = \sum_{\lambda \in Par} e_\lambda(x) \, m_\lambda(y)$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\lambda \in Par_n} e_\lambda(x) \, m_\lambda(y) \right)$$

By prev LEM

$$= \sum_{n \in \mathbb{N}} \left(\sum_{\lambda \in Par_n} s_{\lambda^t}(x) \, s_\lambda(y) \right)$$

$$= \sum_{\lambda \in Par} s_{\lambda^t}(x) \, s_\lambda(y)$$

$\square$

We now bring in the power sym functions p_λ

Recall

$$G(t) = \prod_{i=1}^{\infty} (1 + x_i t) = \sum_{n \in \mathbb{N}} e_n t^n$$

Take derivative of $G(t)$ wrt t

View I:

$$G'(t) = \sum_{n=1}^{\infty} n e_n t^{n-1}$$

View II:

$$G'(t) = \frac{d}{dt} \prod_{i=1}^{\infty} (1 + x_i t)$$

(prod rule)

$$= \sum_{j=1}^{\infty} \underbrace{\left(\prod_{\substack{i=1 \\ i \neq j}}^{\infty} (1 + x_i t) \right)}_{= \frac{G(t)}{1 + x_j t}} \underbrace{\frac{d}{dt} (1 + x_j t)}_{= x_j}$$

$$= G(t) \sum_{j=1}^{\infty} \frac{x_j}{1 + x_j t}$$

$$[t \to -t]$$

$$G'(-t) = G(-t) \sum_{j=1}^{\infty} \frac{x_j}{1-x_j t}$$

$$= G(-t) \sum_{j=1}^{\infty} \left(x_j + x_j^2 t + x_j^3 t^2 + \dots \right)$$

$$= G(-t) \underbrace{\left(p_1 + p_2 t + p_3 t^2 + \dots \right)}_{\substack{\parallel \text{ def} \\ P(t)}}$$

So

$$G'(-t) = G(-t)\, P(t)$$

For $n \in \mathbb{N}$ compare coefs of t^n to get

$$(n+1) e_{n+1} = \sum_{i=0}^{n} (-1)^{n-i} e_i \, p_{n-i+1} \qquad (*)$$

Ex

$$e_1 = p_1$$

$$2e_2 = -p_2 + e_1 p_1$$

$$3e_3 = p_3 - e_1 p_2 + e_2 p_1$$

$$4e_4 = -p_4 + e_1 p_3 - e_2 p_2 + e_3 p_1$$

$$\dots$$

Apply S to each side of $(*)$ to get

$$(n+1)\, h_{n+1} = \sum_{i=0}^{n} h_i\, p_{n-i+1} \qquad n \in \mathbb{N} \qquad (**)$$

Ex

$$h_1 = p_1$$
$$2h_2 = p_2 + h_1 p_1$$
$$3h_3 = p_3 + h_1 p_2 + h_2 p_1$$
$$4h_4 = p_4 + h_1 p_3 + h_2 p_2 + h_3 p_1$$
$$\dots$$

Recall

$$\underbrace{\prod_{i=1}^{\infty} \frac{1}{1-x_i t}}_{= (G(-t))^{-1}} = \underbrace{\sum_{n\in\mathbb{N}} h_n t^n}_{\overset{\text{def}}{=} H(t)}$$

$(**)$ is saying

$$H'(t) = H(t)\, P(t)$$

this identity can be checked directly

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(details)

$$H'(t) = \frac{d}{dt}\left(\prod_{i=1}^{\infty} \frac{1}{1-x_i t}\right)$$

$$= \sum_{j=1}^{\infty} \underbrace{\left(\prod_{\substack{i=1 \\ i\neq j}}^{\infty} \frac{1}{1-x_i t}\right)}_{\substack{\shortparallel \\ H(t)(1-x_j t)}} \frac{d}{dt}\frac{1}{1-x_j t}$$

$$= H(t) \sum_{j=1}^{\infty} \underbrace{(1-x_j t)\frac{d}{dt}\frac{1}{1-x_j t}}_{\substack{\shortparallel \text{ calculus} \\ \frac{x_j}{1-x_j t}}}$$

$$= H(t)\left(\sum_{j=1}^{\infty}\left(x_j + x_j^2 t + \cdots\right)\right)$$

$$= H(t)\,P(t)$$