Lec 21 Monday Oct 24

The coproduct Δ acts on the Schur functions s_λ as follows.

Thm For $\lambda \in Par$

$$\Delta(s_\lambda) = \sum_{\substack{\mu \in Par \\ \mu \subseteq \lambda}} s_\mu \otimes s_{\lambda/\mu}$$

pf Recall variables

x: $x_1, x_2, \ldots$

y: $y_1, y_2, \ldots$

z: $x_1, y_1, x_2, y_2, \ldots$

Recall k-alg hom

$$\theta : R(x) \otimes R(x) \longrightarrow R(z)$$

$$x_i \otimes 1 \longrightarrow x_i$$

$$1 \otimes x_i \longrightarrow y_i$$

We have

$$\Delta_\lambda = \sum_{\substack{T = \text{col strict tableaux} \\ \text{shape } \lambda}} x^{\text{cnt}(T)}$$

$$\Big\downarrow \text{can}$$

$$\sum_{T = \text{CST shp } \lambda} z^{\text{cnt}(T)} \qquad (1)$$

$$\| \, ?$$

$$\sum_{\mu \subseteq \lambda} \theta\left(\Delta_\mu \otimes \Delta_{\lambda/\mu} \right) \qquad (2)$$

Each of (1), (2) is in $\Lambda(z)$

show (1) = (2)

Describe (1)

Write

$$\mathrm{cnt}(T) = (a_1, a_2, a_3, \dots)$$

Define

$$oc(T) = (a_1, a_3, a_5, \dots) \qquad \text{"odd content"}$$

$$ec(T) = (a_2, a_4, a_6, \dots) \qquad \text{"even content"}$$

So

$$z^{\mathrm{cnt}(T)} = x^{oc(T)} \, y^{ec(T)}$$

So (1) equals

$$\sum_{T = \mathrm{SST} \ \mathrm{shp}\ \lambda} x^{oc(T)} \, y^{ec(T)} \qquad (1')$$

Describe (2)

$$(2) = \theta\left(\sum_{\mu \subseteq \lambda}\left(\sum_{\substack{T' = CST \\ \text{shp } \mu}} x^{\text{cnt}(T')}\right) \otimes \left(\sum_{\substack{T'' = CST \\ \text{shp } \lambda/\mu}} x^{\text{cnt}(T'')}\right)\right)$$

$$= \sum_{\mu \subseteq \lambda} \sum_{\substack{T' = CST \\ \text{shp } \mu}} \sum_{\substack{T'' = CST \\ \text{shp } \lambda/\mu}} x^{\text{cnt}(T')} y^{\text{cnt}(T'')} \qquad (2')$$

show $(1') = (2')$

Recall meaning of CST:

- strictly inc down cols
- weakly inc along rows

In above def, no harm in replacing usual order $1 < 2 < 3 < \cdots$

by the order

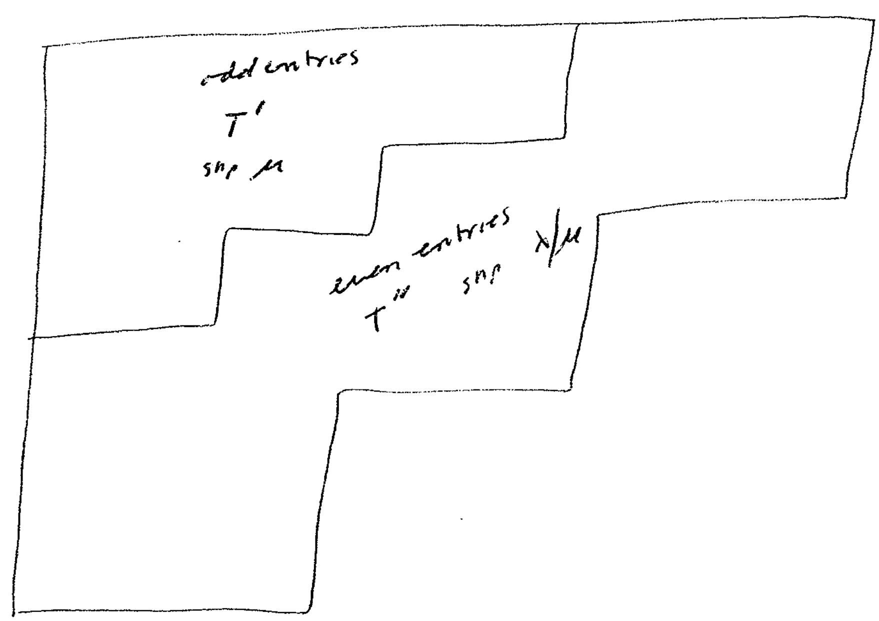
$$\underbrace{1 < 3 < 5 < \cdots}_{\text{odds}} < \underbrace{2 < 4 < 6 < \cdots}_{\text{evens}}.$$

Adopting this point of view we obtain a bijection between the terms in (1'), (2'):

Given a CST T of shape λ

- the boxes in T that contain an odd integer form a CST T' of some shape $\mu \subseteq \lambda$
- the boxes in T that contain an even integer form a CST T'' of skew shape λ/μ
- $x^{oc(T)} = x^{cnt(T')}$
- $y^{ec(T)} = y^{cnt(T'')}$

T: shp λ



the desired bijection sends $T \to \mu, T', T''$ $\square$

Earlier we saw some determinant formula for s_λ

We cite a more general result.

thm For $\lambda, \mu \in Par$

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - i + j - \mu_j} \right)_{1 \le i,j \le l}$$

$$s_{\lambda^t/\mu^t} = \det\left(e_{\lambda_i - i + j - \mu_j} \right)_{1 \le i,j \le l}$$

where l is any integer such that
$l \geq \text{length}(\lambda)$, $l \geq \text{length}(\mu)$ $\square$

pf see notes for citations

COR For $\lambda, \mu \in Par$

(i) $S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}$

(ii) $\omega(s_{\lambda/\mu}) = s_{\lambda^t/\mu^t}$

pf For the 1st equation in the prev thm, apply the k-algebra isomorphisms S, ω to each side and recall

$S(h_\nu) = (-1)^{|\nu|} e_\nu$ $\nu \in Par$

$\omega(h_\nu) = e_\nu$

$\square$

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COR Fn $\lambda \in Par$

(i) $S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda^t}$

(ii) $\omega(s_\lambda) = s_{\lambda^t}$

pf In above LEM take $\mu = \phi$ $\square$

Until further notice, fix $n \in \mathbb{N}$ and write

$$V = \Lambda_n$$

For $\lambda \in \mathrm{Par}_n$ define

$$V_{\trianglelefteq \lambda} = \sum_{\substack{\mu \in \mathrm{Par}_n \\ \mu \trianglelefteq \lambda}} K s_\mu$$

$$V_{\trianglerighteq \lambda} = \sum_{\substack{\mu \in \mathrm{Par}_n \\ \mu \trianglerighteq \lambda}} K s_\mu$$

Obs each of $V_{\trianglelefteq \lambda}$, $V_{\trianglerighteq \lambda}$ is a finite-free K-module, and

$$V_{\trianglelefteq \lambda} \cap V_{\trianglerighteq \lambda} = K s_\lambda$$

LEM For $\lambda \in \mathrm{Par}_n$, each of the following (1)–(3) is a k-basis for $V_{\trianglelefteq \lambda}$:

$$s_\mu \qquad \mu \trianglelefteq \lambda \tag{1}$$

$$m_\mu \qquad \mu \trianglelefteq \lambda \tag{2}$$

$$e_\mu \qquad \mu \trianglerighteq \lambda^c \tag{3}$$

pf Use

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} k^\lambda_\mu \, m_\mu \qquad k^\lambda_\lambda = 1$$

$$e_\lambda = \sum_{\mu \trianglelefteq \lambda^c} a^\lambda_\mu \, m_\mu \qquad a^\lambda_{\lambda^c} = 1$$

$\square$

LEM For $\lambda \in \mathrm{Par}\, n$ each of the following (4)–(6) is a K-basis for $V_{\trianglerighteq \lambda}$:

$$s_\mu \qquad \mu \trianglerighteq \lambda \tag{4}$$

$$w_\mu \qquad \mu \trianglelefteq \lambda^t \tag{5}$$

$$h_\mu \qquad \mu \trianglerighteq \lambda \tag{6}$$

pf Apply the K-alg iso ω to each element in (1)–(3), and use

$$\omega(s_\mu) = s_{\mu^t}$$

$$\omega(e_\mu) = h_\mu$$

$$\omega(m_\mu) = w_\mu$$

$\square$

Thm For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$

(i) the intersection

$$\left(\sum_{\mu \trianglerighteq \lambda} \Bbbk h_\mu \right) \cap \left(\sum_{\nu \trianglerighteq \lambda^t} \Bbbk e_\nu \right) \qquad (*)$$

is a finite-free $\Bbbk$-module with basis Δ_λ.

(ii) Δ_λ is the unique element of $(*)$ with h_λ-coef 1

(iii) Δ_λ is the unique element of $(*)$ with e_{λ^t}-coef 1

pf (i) Obs

$$(*) = V_{\trianglerighteq\lambda} \cap V_{\trianglelefteq\lambda} = \Bbbk \Delta_\lambda$$

(iii) Since $K_\lambda^\lambda = 1 = a_{\lambda^t}^\lambda$

(ii) Apply ω to everything in (iii) $\square$

Next goal:

For variables

$x: \quad x_1, x_2, \dots$

$y: \quad y_1, y_2, \dots$

show

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-x_i y_j} = \sum_{\lambda \in Par} s_\lambda(x)\, s_\lambda(y)$$

LEM We have

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-x_i y_j} = \sum_{\lambda \in Par} h_\lambda(x)\, m_\lambda(y) \qquad (*)$$

$$= \sum_{\lambda \in Par} m_\lambda(x)\, h_\lambda(y) \qquad (**)$$

pf $(*)$ Recall

$$\prod_{i=1}^{\infty} \frac{1}{1-x_i t} = \sum_{n\in\mathbb{N}} h_n t^n$$

So

$$\prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-x_i y_j} = \prod_{j=1}^{\infty} \left(\prod_{i=1}^{\infty} \frac{1}{1-x_i y_j} \right)$$

$$= \prod_{j=1}^{\infty} \left(\sum_{n\in\mathbb{N}} h_n(x) y_j^n \right)$$

Fr $\lambda \in$ Par, the coeff of $h_\lambda(x)$ above is

$$\left[y_1^{\lambda_1} y_2^{\lambda_2} \cdots \right]$$

which is $m_\lambda(y)$

Result follows.

$(**)$ Sim. $\square$

LEM For $n \in \mathbb{N}$,

$$\sum_{\lambda \in \mathrm{Par}_n} h_\lambda \otimes m_\lambda = \sum_{\lambda \in \mathrm{Par}_n} m_\lambda \otimes h_\lambda$$

pf By prev lem,

$$\sum_{\lambda \in \mathrm{Par}_n} h_\lambda(x)\, m_\lambda(y) = \sum_{\lambda \in \mathrm{Par}_n} m_\lambda(x)\, h_\lambda(y)$$

Result follows. $\square$

Fix $n \in \mathbb{N}$ and write $V = \Lambda_n$

Recall $V_{\trianglerighteq \lambda}$, $V_{\trianglelefteq \lambda}$ $\lambda \in Par_n$

Define

$$V^+ = \sum_{\lambda \in Par_n} V_{\trianglerighteq \lambda} \otimes V_{\trianglelefteq \lambda}$$

$$V^- = \sum_{\lambda \in Par_n} V_{\trianglelefteq \lambda} \otimes V_{\trianglerighteq \lambda}$$

LEM We have

(i) V^+ is a finite-free K-module with K-basis

$s_\mu \otimes s_\nu$ $\mu, \nu \in Par_n$, $\mu \trianglerighteq \nu$ (*)

(ii) V^- is a finite-free K-module with K-basis

$s_\mu \otimes s_\nu$ $\mu, \nu \in Par_n$, $\mu \trianglelefteq \nu$

(iii) $V^+ \cap V^-$ is a finite-free K-module with K-basis

$s_\mu \otimes s_\mu$ $\mu \in Par_n$

pf (i) One checks $(*) \subseteq V^+$

show $V^+ \subseteq \mathrm{Span}_K(*)$:

For $v \in V_{\trianglerighteq \lambda}$ and $w \in V_{\trianglelefteq \lambda}$

show $v \otimes w \in \mathrm{Span}_K(*)$

Write

$$v = \sum_{\mu \trianglerighteq \lambda} a_\mu s_\mu$$

$$w = \sum_{\nu \trianglelefteq \lambda} b_\nu s_\nu$$

$$v \otimes w = \sum_{\nu \trianglelefteq \lambda \trianglelefteq \mu} a_\mu b_\nu \, s_\mu \otimes s_\nu$$

$$\in \sum_{\nu \trianglelefteq \mu} K \, s_\mu \otimes s_\nu$$

$$= \mathrm{Span}_K(*)$$

(ii) Similar

(iii) By (i), (ii) $\square$