Lec 20 Friday Oct 21

A Combinatorial approach to Schur functions

<u>Motivation</u> For $\lambda \in Par$ write s_λ

in terms of monomial sym functions m_μ

ex $|\lambda| = 3$

$$s_{(3)} = h_3 = m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$$

$$s_{(2,1)} = \begin{vmatrix} e_2 & e_3 \\ 1 & e_1 \end{vmatrix} = e_2 e_1 - e_3 = m_{(2,1)} + 2m_{(1,1,1)}$$

$$s_{(1,1,1)} = e_3 = m_{(1,1,1)}$$

	$s_{(1,1,1)}$	$s_{(2,1)}$	$s_{(3)}$
$m_{(1,1,1)}$	1	2	1
$m_{(2,1)}$	0	1	1
$m_{(3)}$	0	0	1

The entries in above matrix are nonneg integers

We now consider what do they count

DEF For $\lambda \in$ Par, a

<u>column strict tableaux of shape λ</u> is

an assignment T of a pos integer to each box of a Young diagram of shape λ, such that

- strictly inc down each column
- weakly inc along each row

For a column strict tableaux T

the <u>content</u> $\mathrm{cont}(T)$ is the weak composition

$a_1, a_2, \ldots$

where

$a_i = \#$ boxes containing i

the <u>corresp monomial</u> is

$$X^{\mathrm{cont}(T)} = x_1^{a_1} x_2^{a_2} \cdots$$

We now officially define the Schur functions

DEF For $\lambda \in Par$ let

$$s_\lambda = \sum_T x^{\text{cont}(T)}$$

sum is over all column strict tableaux T of shape λ

"Schur function of type λ"

Ex $|\lambda| = 3$

$\lambda = 3$

poss T :

i	j	k

$i \le j \le k$

corresp monomial: $x_i x_j x_k$

$$s_\lambda = \sum_{i \le j \le k} x_i x_j x_k$$

$$= h_3$$

$$= m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$$

$\lambda = 2,1$

$$\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

poss T		corresp monomial
$\begin{array}{\|c\|c\|}\hline i & i \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$	$i<j$	$x_i^2 x_j$
$\begin{array}{\|c\|c\|}\hline i & j \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$	$i<j$	$x_i x_j^2$
$\begin{array}{\|c\|c\|}\hline i & j \\ \hline k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$	$i<j<k$	$x_i x_j x_k$
$\begin{array}{\|c\|c\|}\hline i & k \\ \hline j & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$		$x_i x_j x_k$

$$A_\lambda = \sum_{i \neq j} x_i^2 x_j + 2 \sum_{i<j<k} x_i x_j x_k$$

$$= m_{21} + 2\, m_{111}$$

$\lambda = 1,1,1$ $\qquad \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array}$

poss T $\qquad \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline \end{array}$ $\qquad i<j<k$

$x_i x_j x_k$

corresp monomial

$$s_\lambda = \sum_{i<j<k} x_i x_j x_k$$

$$= m_{1,1,1}$$

$\square$

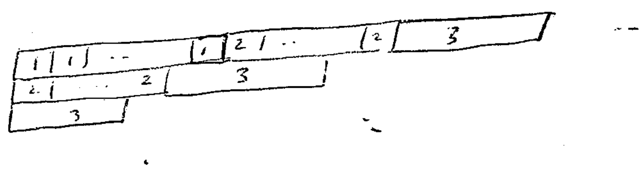
Prop $s_\lambda \in \Lambda$ for $\lambda \in \mathrm{Par}$

pf Recall the $S_{(\infty)}$ action on $x_1, x_2, \dots$

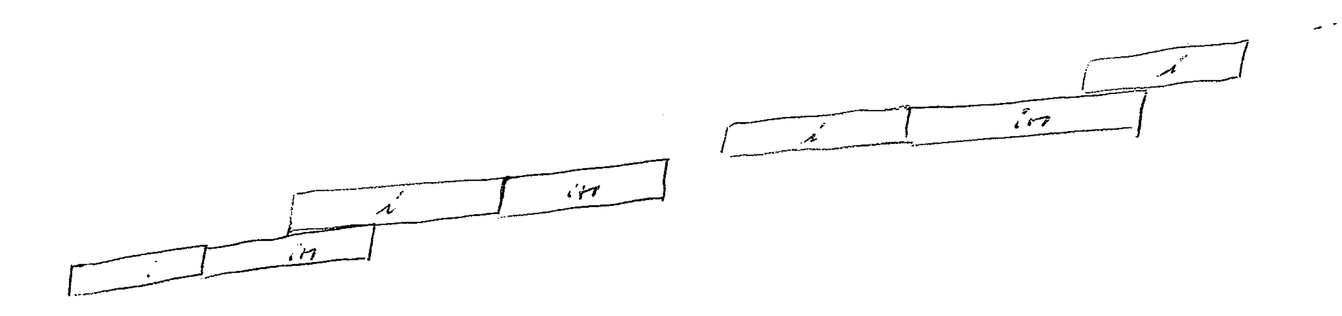
For $i = 1, 2, \dots$ show s_λ is invar under the swap $\sigma : x_i \leftrightarrow x_{i+1}$

Consider a col strict tableaux T shape λ

T:



Consider location of boxes containing $i, i+1$:



let $a = \#$ boxes containing i

$b = \quad$ -- $\quad i{+}1$

so $X^{\text{cont}(T)} = \cdots x_i^a \, x_{i+1}^b \cdots$

We now construct a col strict tableaux T' of shape λ as follows:

For each box in T, call it <u>in play</u> if it contains i or $i{+}1$

For a box in play, call it <u>immobile</u> whenever its column contains another box in play

Call it <u>mobile</u> else

For each row of T consider the mobile boxes in play: $\quad *$

$$\underbrace{\boxed{i}\,\boxed{i}\,\cdots\,\boxed{i}}_{r}\ \underbrace{\boxed{i{+}1}\,\boxed{i{+}1}\,\cdots\,\boxed{i{+}1}}_{s}$$

To get T' from T

- leave alone all boxes except the mobile ones
- adjust the entries in the mobile boxes so each row $*$ becomes

$$\underbrace{\boxed{i}\,\boxed{i}\,\cdots\,\boxed{i}}_{s}\ \underbrace{\boxed{i{+}1}\,\boxed{i{+}1}\,\cdots\,\boxed{i{+}1}}_{r}$$

One checks the move $T \to T'$

is an involution st

$$\sigma\left(x^{\mathrm{cont}(T)}\right) = x^{\mathrm{cont}(T')}$$

Therefore σ leaves s_λ invar. $\square$

For $n \in \mathbb{N}$ recall a $\mathbb{K}$-basis for Λ_n:

$$m_\lambda \qquad \lambda \in \mathrm{Par}_n$$

For $\lambda \in \mathrm{Par}_n$ write

$$s_\lambda = \sum_{\mu \in \mathrm{Par}_n} K^\lambda_\mu \, m_\mu \qquad K^\lambda_\mu \in \mathbb{K}$$

LEM For $\lambda, \mu \in \mathrm{Par}_n$

K^λ_μ = # col strict tableaux that have shape λ and content μ

" Kostka number for λ, μ "

pf By constr $\square$

LEM For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$

(i) $K^\lambda_\mu = 1$ if $\lambda = \mu$

(ii) $K^\lambda_\mu = 0$ unless $\lambda \trianglerighteq \mu$

pf (i) For the only col strict tableaux of shape λ and content λ,

In row 1 each box contains 1
-- 2 ... 2
...

(ii) Assume $K^\lambda_\mu \neq 0$. Show $\lambda \trianglerighteq \mu$

$\exists$ col strict tableaux T of shape λ and content μ.

For $i = 1, 2, ..$

T has $\mu_1 + \mu_2 + \cdots + \mu_i$ boxes that contain an entry $1, 2, .., i$ and these boxes are all in rows at most i,

So $\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i$

$\square$

COR For $n \in \mathbb{N}$ the following is a K-basis for Λ_n:

$$\Delta_\lambda \qquad \lambda \in \mathrm{Par}_n.$$

$\square$

— ∘ —

Next goal: For $\lambda \in \mathrm{Par}$ find $\Delta(e_\lambda)$

DEF For $\lambda, \mu \in \mathrm{Par}$ declare

$\mu \subseteq \lambda$ whenever $\mu_i \le \lambda_i$ for $i = 1, 2, \dots$

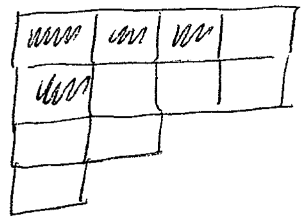
"Young diag μ fits inside Young diag λ"

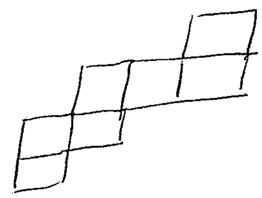
Assume $\mu \subseteq \lambda$

To get <u>skew diagram</u> λ/μ, delete from λ the boxes of μ

Ex $\lambda = 4, 4, 2, 1$

$\mu = 3, 1$



λ/μ : 

DEF For a skew diag λ/μ a col strict tableaux of shape λ/μ is an assignment T of a pos integer to each box in λ/μ s.t

- strictly inc down each col
- weakly inc along each row

For such T, the content $cnt(T)$ is defined as before

DEF For a skew diag λ/μ define

$$s_{\lambda/\mu} = \sum_{T} x^{\mathrm{cnt}(T)}$$

sum over all col strict tableaux of shape λ/μ.

For $\lambda, \mu \in \mathrm{Par}$ st $\mu \not\subseteq \lambda$ declare

$$s_{\lambda/\mu} = 0.$$

LEM. For a skew diag λ/μ.

$$s_{\lambda/\mu} \in \Lambda$$

Moreover

$$s_{\lambda/\mu} \in \Lambda_n$$

where $n = |\lambda/\mu| = \#$ boxes in λ/μ
$= |\lambda| - |\mu|$

$\square$

pf ex