

Lec 2 Friday Sept 9

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We continue to discuss coalgebras

Ex K becomes a K -coalgebra with

$$\Delta: K \rightarrow K \otimes K$$
$$1 \rightarrow 1 \otimes 1$$

$$\varepsilon: K \rightarrow K$$
$$1 \rightarrow 1$$

"trivial coalgebra"

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More examples shortly.

REV Given K -modules U, V, U', V'
and K -module homs

$$\varphi: U \rightarrow V \qquad \varphi': U' \rightarrow V'$$

Then \exists K -module hom

$$\begin{aligned} \varphi \otimes \varphi': U \otimes U' &\rightarrow V \otimes V' \\ u \otimes u' &\rightarrow \varphi(u) \otimes \varphi'(u') \end{aligned}$$

Check $\varphi \otimes \varphi'$ respects scalar mult:

$$\begin{aligned} \varphi \otimes \varphi' (\underbrace{\alpha (u \otimes u')}_{(\alpha u) \otimes u'}) &\stackrel{?}{=} \underbrace{\alpha (\varphi \otimes \varphi' (u \otimes u'))}_{\varphi(u) \otimes \varphi'(u')} \\ \underbrace{\varphi(\alpha u) \otimes \varphi'(u')}_{\alpha \varphi(u)} & \qquad \text{OK} \\ \alpha (\varphi(u) \otimes \varphi'(u')) & \end{aligned}$$

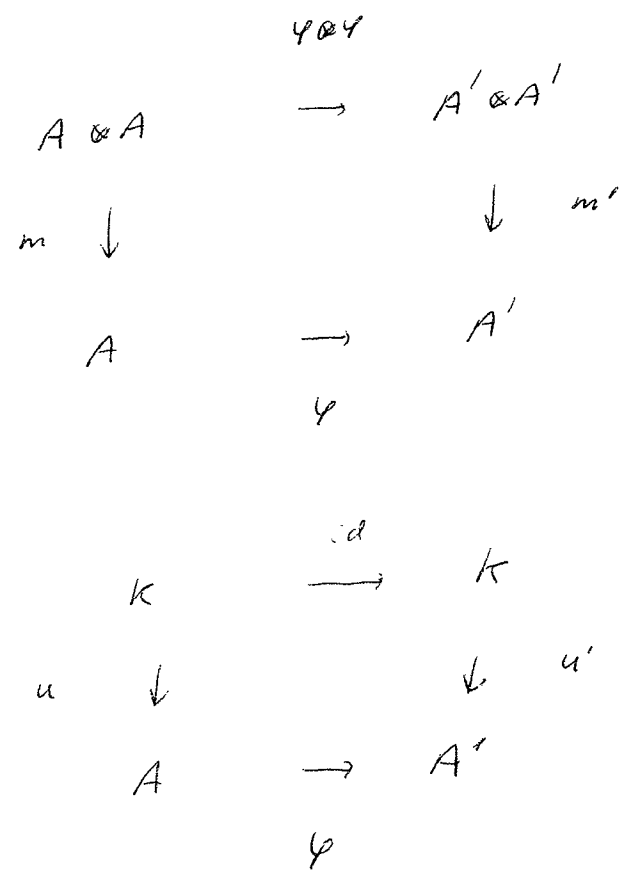
REV Given k -algebras A, A'

a map $\varphi: A \rightarrow A'$ is a k -alg homomorphism
"alg morphism"

whenever

- φ is k -module hom
- $\varphi(ab) = \varphi(a)\varphi(b) \quad a, b \in A$
- $\varphi(1_A) = 1_{A'}$

In other words, φ is k -module hom and



REV Given K -algebras A, B, A', B'

and K -alg morphisms

$$\varphi: A \rightarrow B, \quad \varphi': A' \rightarrow B'$$

then the map

$$\begin{aligned} \varphi \otimes \varphi': A \otimes A' &\rightarrow B \otimes B' \\ a \otimes a' &\rightarrow \varphi(a) \otimes \varphi'(a') \end{aligned}$$

is a K -alg morphism

pf $\varphi \otimes \varphi'$ is K -module hom

check $\varphi \otimes \varphi' \left(\underbrace{a \otimes a' \quad a_i \otimes a'_i}_{aa_i \otimes a'_i a'_i} \right) \stackrel{?}{=} \left(\varphi \otimes \varphi' (a \otimes a') \right) \left(\varphi \otimes \varphi' (a_i \otimes a'_i) \right)$

$$\underbrace{\varphi(aa_i) \otimes \varphi'(a'_i a'_i)}_{\varphi(a)\varphi(a_i) \quad \varphi'(a'_i)\varphi'(a'_i)} = \underbrace{\varphi(a) \otimes \varphi'(a'_i)}_{\varphi(a)\varphi(a_i) \quad \varphi'(a'_i)\varphi'(a'_i)} \left(\varphi \otimes \varphi' (a_i \otimes a'_i) \right)$$

OK

check $\varphi \otimes \varphi' (1_{A \otimes A'}) \stackrel{?}{=} 1_{B \otimes B'}$

$$\underbrace{\varphi \otimes \varphi' (1_{A \otimes A'})}_{\varphi(1_A) \otimes \varphi'(1_{A'})} \stackrel{?}{=} 1_{B \otimes B'}$$

$$\varphi(1_A) \otimes \varphi'(1_{A'}) \stackrel{OK}{=} 1_B \otimes 1_{B'}$$

LEM Given K -module V .

Then the tensor algebra $T(V)$ becomes a K -coalgebra
with Δ, ε as follows

(i) $\Delta : T(V) \rightarrow T(V) \otimes T(V)$ is a K -alg hom
that sends $v \rightarrow v \otimes 1 + 1 \otimes v$
 $\forall v \in V$

(ii) $\varepsilon : T(V) \rightarrow K$ is a K -alg hom that sends
 $v \rightarrow 0$ $\forall v \in V$

pf the map

$$\begin{aligned} V &\rightarrow T(V) \otimes T(V) & (*) \\ v &\rightarrow v \otimes 1 + 1 \otimes v \end{aligned}$$

is K -linear. So (*) extends to

a K -alg hom

$$\Delta : T(V) \rightarrow T(V) \otimes T(V)$$

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by the universal property of $T(V)$

check Δ makes diagram commute:

$$\begin{array}{ccc} & \Delta \otimes \text{id} & \\ & \leftarrow C \otimes C & \\ C \otimes C \otimes C & & \\ \text{id} \otimes \Delta \uparrow & & \uparrow \Delta \\ C \otimes C & \leftarrow C & \\ & \Delta & \end{array}$$

All the maps involved are K -alg homs

WLOG chase generators $v \in V$ around diag

$$\begin{array}{ccc}
 (v \otimes 1 + 1 \otimes v) \otimes 1 + (1 \otimes 1) \otimes v & \leftarrow & v \otimes 1 + 1 \otimes v \\
 \parallel & & \\
 v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v & & \\
 \parallel & & \uparrow \\
 v \otimes (1 \otimes 1) + 1 \otimes (v \otimes 1 + 1 \otimes v) & & \\
 \uparrow & & \\
 v \otimes 1 + 1 \otimes v & \leftarrow & v
 \end{array}$$

The map

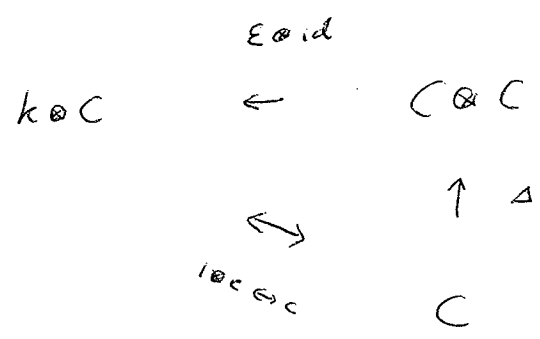
$$\begin{array}{l}
 V \rightarrow k \\
 v \rightarrow 0
 \end{array}$$

(**)

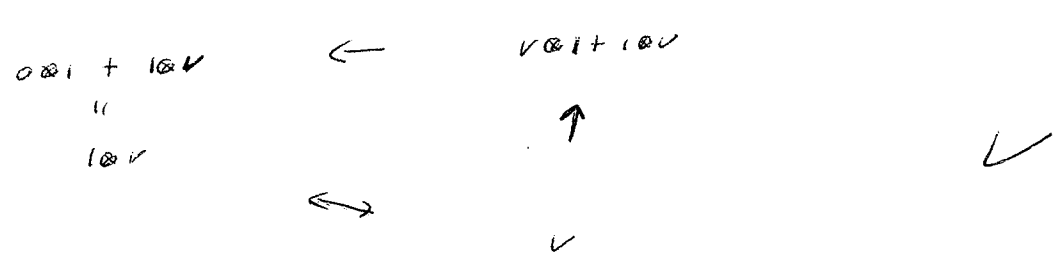
is k -linear,
 so (**) extends to k -alg hom

$$\varepsilon: T(V) \rightarrow k$$

check ε makes diag commute



All maps involved are k -alg homs
 Chase gen $v \in V$ around diag:



□

LEM Given K -module V
 the symmetric alg $S(V)$ becomes a
 K -coalgebra with Δ, ε as follows

- (i) $\Delta : S(V) \rightarrow S(V) \otimes S(V)$ is K -alg hom
 that sends $v \rightarrow v \otimes 1 + 1 \otimes v \quad \forall v \in V$
- (ii) $\varepsilon : S(V) \rightarrow K$ is K -alg hom that sends
 $v \rightarrow 0 \quad \forall v \in V$

pf Recall 2-sided ideal J of $T(V)$ gen by
 $uv - vu \quad u, v \in V$

For K -alg hom

$$\Delta : T(V) \rightarrow T(V) \otimes T(V)$$

apply Δ to J :

$\forall u, v \in V$

$$\begin{aligned} \Delta(uv - vu) &= \Delta(u)\Delta(v) - \Delta(v)\Delta(u) \\ &= (u \otimes 1 + 1 \otimes u) \otimes (v \otimes 1 + 1 \otimes v) \\ &\quad - (v \otimes 1 + 1 \otimes v) \otimes (u \otimes 1 + 1 \otimes u) \\ &= (uv - vu) \otimes 1 + 1 \otimes (uv - vu) \end{aligned}$$

$$\text{So } \Delta(J) \subseteq J \otimes T(V) + T(V) \otimes J$$

Consider composition

$$T(V) \xrightarrow{\quad} T(V) \otimes T(V) \xrightarrow{\text{can} \otimes \text{can}} S(V) \otimes S(V) \quad *$$

$$x \otimes y \quad \rightarrow \quad (x+J) \otimes (y+J)$$

$$J \otimes T(V) + T(V) \otimes J \quad \rightarrow \quad 0$$

So (*) sends $J \rightarrow 0$

So (*) induces k -alg morphism

$$\Delta: S(V) \rightarrow S(V) \otimes S(V)$$

that sends

$$v \rightarrow v \otimes 1 + 1 \otimes v \quad \forall v \in V$$

For k -alg morphism

$$\varepsilon: T(V) \rightarrow k$$

apply ε to J :

$\forall u, v \in V$

$$\begin{aligned} \varepsilon(uv - vu) &= \varepsilon(u)\varepsilon(v) - \varepsilon(v)\varepsilon(u) \\ &\quad \parallel \quad \parallel \quad \parallel \quad \parallel \\ &\quad 0 \quad 0 \quad 0 \quad 0 \\ &= 0 \end{aligned}$$

so ε induces k -alg hom

$$\varepsilon: S(V) \rightarrow k$$

that sends $v \rightarrow 0 \quad \forall v \in V$

One checks the required diagrams commute.



LEM Given a group G
 the group alg KG becomes a K -alg with

$$\Delta : \begin{matrix} C \\ \parallel \\ KG \end{matrix} \rightarrow KG \otimes KG$$

$$t_g \rightarrow t_g \otimes t_g$$

$$\varepsilon : \begin{matrix} KG \rightarrow K \\ t_g \rightarrow 1 \end{matrix}$$

pf Above Δ, ε exist as K -module homs
 check diagrams

$$\begin{matrix} C \otimes C \otimes C \otimes C \leftarrow C \otimes C \\ \uparrow \quad \uparrow \\ C \otimes C \leftarrow C \end{matrix}$$

$$\begin{matrix} (t_g \otimes t_g) \otimes t_g \leftarrow t_g \otimes t_g \\ \uparrow \\ t_g \otimes (t_g \otimes t_g) \\ \uparrow \\ t_g \otimes t_g \leftarrow t_g \end{matrix} \quad \checkmark$$

$$\begin{matrix} K \otimes C \leftarrow C \otimes C \\ \searrow \quad \uparrow \\ C \end{matrix}$$

$$\begin{matrix} 1 \otimes t_g \leftarrow t_g \otimes t_g \\ \searrow \quad \uparrow \\ t_g \end{matrix}$$

□

Given coalgebras C, C'

Lets turn $C \otimes C'$ into coalg.
 \parallel
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$C \otimes C'$ is K -module \checkmark

Desire K -module hom

$$\Delta : C \otimes C' \longrightarrow C \otimes C' \otimes C \otimes C'$$

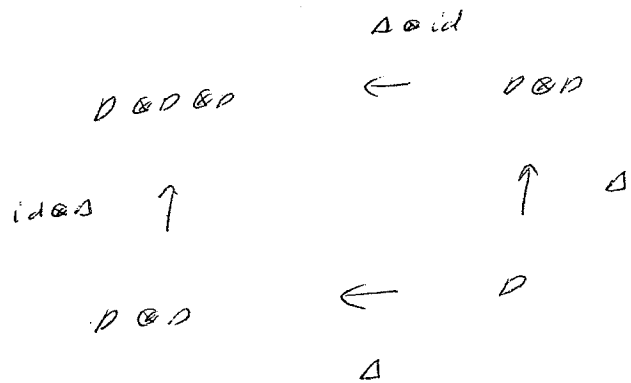
Have

$$C \otimes C' \xrightarrow{\Delta \otimes \Delta'} C \otimes C \otimes C' \otimes C'$$

Guess

$$\Delta : C \otimes C' \xrightarrow{\Delta \otimes \Delta'} C \otimes C \otimes C' \otimes C' \xrightarrow{id \otimes \tau_{C,C'} \otimes id} C \otimes C' \otimes C \otimes C' \quad (\star)$$

check if (\star) makes this commute:



For $c \in C$ and $c' \in C'$

chase $c \otimes c' \in D$ around diag.

under either path, final image is

$$\sum_{(C)} \sum_{(C')} c_1 \otimes c'_1 \otimes c_2 \otimes c'_2 \otimes c_3 \otimes c'_3$$

So diag commutes \checkmark

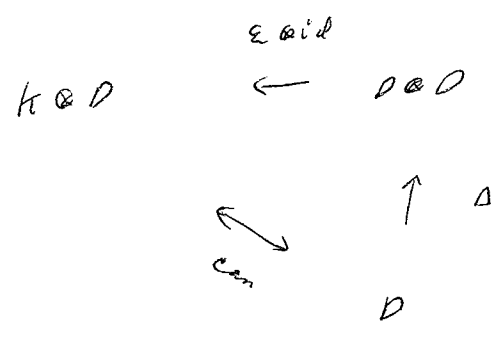
Also desire k -module hom

$$\epsilon: E \otimes C' \rightarrow K$$

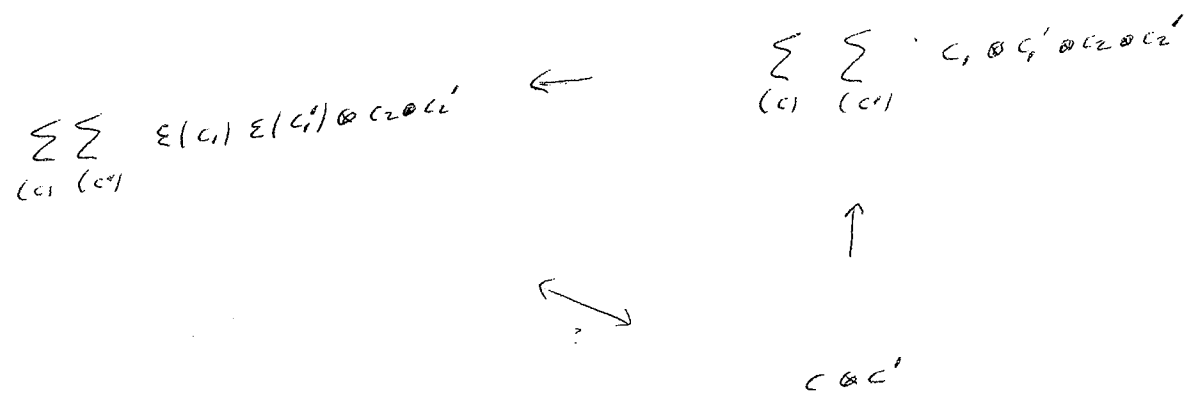
Guess

$$\begin{array}{ccccc}
 E: & C \otimes C' & \longrightarrow & K \otimes K & \longrightarrow & K & (\star\star) \\
 & & & \epsilon \otimes \epsilon' & \xrightarrow{\text{id} \otimes \epsilon'} & &
 \end{array}$$

check $(\star\star)$ makes this diag commute:



For $c \in C, c' \in C'$ chase $c \otimes c' \in D$ around diag



Require

$$\sum_{(c_1)} \sum_{(c'_1)} \varepsilon(c_1) \varepsilon(c'_1) c_2 \otimes c'_2 = c \otimes c'$$

$$\sum_{(c_1)} \sum_{(c'_1)} \varepsilon(c_1) c_2 \otimes \varepsilon(c'_1) c'_2$$

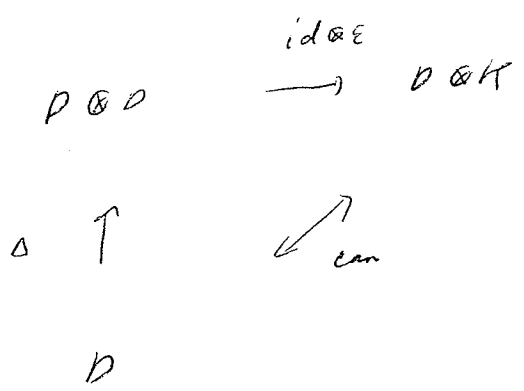
$$\left(\sum_{(c_1)} \varepsilon(c_1) c_2 \right) \otimes \left(\sum_{(c'_1)} \varepsilon(c'_1) c'_2 \right)$$

" " " "

C C'

OK

Similarly $(\star \star)$ makes this diag commute



We have turned $C \otimes C'$ into a coalg □