Lec 2 Friday Sept 9

We continue to discuss coalgebras

Ex k becomes a k-coalgebra with

$$\Delta : \begin{array}{l} k \longrightarrow k \otimes k \\ 1 \longrightarrow 1 \otimes 1 \end{array}$$

$$\varepsilon : \begin{array}{l} k \longrightarrow k \\ 1 \to 1 \end{array}$$

"trivial coalgebra"

— o —

More examples shortly.

REV Given K-modules U, V, U', V'

and K-module homs

$\varphi: U \to V \qquad \varphi': U' \to V'$

Then $\exists$ K-module hom

$$\varphi \otimes \varphi' : U \otimes U' \to V \otimes V'$$
$$u \otimes u' \to \varphi(u) \otimes \varphi'(u')$$

Check $\varphi \otimes \varphi'$ respects scalar mult:

$$\underbrace{\varphi\otimes\varphi'\big(\underbrace{\alpha(u\otimes u')}_{(\alpha u)\otimes u'}\big)}_{\underbrace{\varphi(\alpha u)}_{\alpha\varphi(u)} \otimes \varphi'(u')} \overset{?}{=} \alpha\big(\underbrace{\varphi\otimes\varphi'(u\otimes u')}_{\varphi(u)\otimes\varphi'(u')}\big)$$

$$\underbrace{\alpha\varphi(u) \otimes \varphi'(u')}_{=\ \alpha\left(\varphi(u)\otimes\varphi'(u')\right)}$$

OK

REV. Given k-algebras A, A'

a map $\varphi: A \to A'$ is a k-alg homomorphism
"alg morphism"

whenever

- φ is k module hom
- $\varphi(ab) = \varphi(a)\varphi(b)$ $\quad a, b \in A$
- $\varphi(1_A) = 1_{A'}$

In other words. φ is k module hom and

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\varphi \otimes \varphi} & A' \otimes A' \\ m \downarrow & & \downarrow m' \\ A & \xrightarrow[\varphi]{} & A' \end{array}$$

$$\begin{array}{ccc} k & \xrightarrow{id} & k \\ u \downarrow & & \downarrow u' \\ A & \xrightarrow[\varphi]{} & A' \end{array}$$

REV Given k-algebras A, B, A', B'

and k-alg morphisms

$$\varphi: A \to B, \qquad \varphi': A' \to B'$$

then the map

$$\varphi \otimes \varphi' : A \otimes A' \to B \otimes B'$$
$$a \otimes a' \to \varphi(a) \otimes \varphi'(a')$$

is a k-alg morphism

pf $\varphi \otimes \varphi'$ is k module hom ✓

check

$$\underbrace{\varphi \otimes \varphi' \Big(\underbrace{a \otimes a' \quad a_1 \otimes a_1'}_{aa_1 \otimes a'a_1'} \Big)}_{\substack{\varphi(aa_1) \otimes \varphi'(a'a_1') \\ \varphi(a)\varphi(a_1) \quad \varphi'(a')\varphi'(a_1')}} \overset{?}{=} \underbrace{\Big(\underbrace{\varphi \otimes \varphi' (a \otimes a')}_{= \varphi(a) \otimes \varphi'(a')} \Big) \Big(\underbrace{\varphi \otimes \varphi' (a_1 \otimes a_1')}_{= \varphi(a_1) \otimes \varphi'(a_1')} \Big)}_{\varphi(a)\varphi(a_1) \otimes \varphi'(a')\varphi'(a_1')}$$

ok

check

$$\underbrace{\varphi \otimes \varphi' (1_{A \otimes A'})}_{\substack{= 1_A \otimes 1_{A'} \\ \varphi(1_A) \otimes \varphi'(1_{A'}) \\ = 1_B \quad = 1_{B'}}} \overset{?}{=} 1_{B \otimes B'} = 1_B \otimes 1_{B'}$$

ok

LEM Given K-module V.

Then the tensor algebra $T(V)$ becomes a K-coalgebra

with Δ, ε as follows

(i) $\Delta : T(V) \to T(V) \otimes T(V)$ is a K-alg hom

that sends $v \longrightarrow v \otimes 1 + 1 \otimes v$

$\forall v \in V$

(ii) $\varepsilon : T(V) \to K$ is a K-alg hom that sends

$v \to 0$ $\forall v \in V$

pf the map

$$V \longrightarrow T(V) \otimes T(V) \qquad (*)$$

$$v \longrightarrow v \otimes 1 + 1 \otimes v$$

is k-linear. So $(*)$ extends to

a k-alg hom

$$\Delta : \underset{\overset{\shortparallel}{C}}{T(V)} \to T(V) \otimes T(V)$$

by the universal property of $T(V)$

Check Δ makes diagram commute:

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes id"'] \\
C \otimes C \arrow[u, "id \otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

All the maps involved are k-alg homs

WLOG chase generator $v \in V$ around diag

$$\begin{array}{ccc}
(v\otimes 1+1\otimes v)\otimes 1 + (1\otimes 1)\otimes v & \longleftarrow & v\otimes 1 + 1\otimes v \\
\shortparallel & & \\
v\otimes 1\otimes 1 + 1\otimes v\otimes 1 + 1\otimes 1\otimes v & & \uparrow \\
\shortparallel & & \\
v\otimes(1\otimes 1) + 1\otimes(v\otimes 1+1\otimes v) & & \\
\uparrow & & \\
v\otimes 1 + 1\otimes v & \longleftarrow & v
\end{array}$$

The map

$$\begin{aligned} V &\longrightarrow k \\ v &\longrightarrow 0 \end{aligned} \qquad (**)$$

is k-linear,
so $(**)$ extends to k-alg hom

$$\varepsilon : T(V) \to k$$

check ε makes diag commute

$$
\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \\
 & C \arrow[u, "\Delta"'] \arrow[ul, leftrightarrow, "1 \otimes c \leftrightarrow c"]
\end{tikzcd}
$$

All maps involved are k-alg homs

Chase gen $v \in V$ around diag:

$$
\begin{tikzcd}
0 \otimes 1 + 1 \otimes v \arrow[d, equal] & v \otimes 1 + 1 \otimes v \arrow[l] \\
1 \otimes v & \\
 & v \arrow[uu] \arrow[ul, leftrightarrow]
\end{tikzcd}
$$

✓

$\square$

LEM Given K-module V

the symmetric alg $S(V)$ becomes a K-coalgebra with Δ, ε as follows

(i) $\Delta : S(V) \to S(V) \otimes S(V)$ is K-alg hom that sends $v \to v \otimes 1 + 1 \otimes v$ $\forall v \in V$

(ii) $\varepsilon : S(V) \to K$ is K-alg hom that sends $v \to 0$ $\forall v \in V$

pf Recall 2-sided ideal J of $T(V)$ gen by $uv - vu$ $u, v \in V$

For K-alg hom

$$\Delta : T(V) \to T(V) \otimes T(V)$$

apply Δ to J:

$\forall$ $u, v \in V$

$$\begin{aligned}
\Delta(uv - vu) &= \Delta(u)\Delta(v) - \Delta(v)\Delta(u) \\
&= (u \otimes 1 + 1 \otimes u)(v \otimes 1 + 1 \otimes v) \\
&\quad - (v \otimes 1 + 1 \otimes v)(u \otimes 1 + 1 \otimes u) \\
&= (uv - vu) \otimes 1 + 1 \otimes (uv - vu)
\end{aligned}$$

So $\Delta(J) \subseteq J \otimes T(V) + T(V) \otimes J$

Consider composition

$$T(V) \longrightarrow T(V) \otimes T(V) \xrightarrow{\text{can} \otimes \text{can}} S(V) \otimes S(V) \qquad *$$

$$x \otimes y \longrightarrow (x+J) \otimes (y+J)$$

$$J \otimes T(V) + T(V) \otimes J \longrightarrow 0$$

So $(*)$ sends $J \to 0$

So $(*)$ induces K alg morphism

$$\Delta : S(V) \to S(V) \otimes S(V)$$

that sends

$$v \longrightarrow v \otimes 1 + 1 \otimes v \qquad \forall v \in V$$

For K-alg morphism

$$\varepsilon : T(V) \to K$$

apply ε to J:

$\forall\, u, v \in V$

$$\varepsilon(uv - vu) = \underset{0}{\underset{\|}{\varepsilon(u)}}\,\underset{0}{\underset{\|}{\varepsilon(v)}} - \underset{0}{\underset{\|}{\varepsilon(v)}}\,\underset{0}{\underset{\|}{\varepsilon(u)}}$$

$$= 0$$

So ε induces k-alg hom

$$\varepsilon : S(V) \to k$$

that sends $v \to 0 \quad \forall v \in V$

One checks the required diagrams commute. $\square$

LEM Given a group G the group alg kG becomes a k-coalg with

$$\Delta : \begin{array}{ccc} C \\ \shortparallel \\ kG & \to & kG \otimes kG \\ t_g & \to & t_g \otimes t_g \end{array}$$

$$\varepsilon : \begin{array}{ccc} kG & \to & k \\ t_g & \to & 1 \end{array}$$

pf Above Δ, ε exist as k-module homs

check diagrams

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l] \\ C \otimes C \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} (t_g \otimes t_g) \otimes t_g = t_g \otimes (t_g \otimes t_g) & t_g \otimes t_g \arrow[l] \\ t_g \otimes t_g \arrow[u] & t_g \arrow[l] \arrow[u] \end{tikzcd} \quad \checkmark$$

$$\begin{tikzcd} k \otimes C & C \otimes C \arrow[l] \\ & C \arrow[u] \arrow[ul, leftrightarrow] \end{tikzcd} \qquad \begin{tikzcd} 1 \otimes t_g & t_g \otimes t_g \arrow[l] \\ & t_g \arrow[u] \arrow[ul, leftrightarrow] \end{tikzcd}$$

$\square$

Given coalgebras C, C'

Lets turn $\begin{array}{c} C \otimes C' \\ \shortparallel \\ D \end{array}$ into coalg.

$C \otimes C'$ is K-module

Desire K-module hom

$$\Delta : C \otimes C' \longrightarrow C \otimes C' \otimes C \otimes C'$$

Have

$$C \otimes C' \xrightarrow[\Delta \otimes \Delta']{} C \otimes C \otimes C' \otimes C'$$

Guess

$$\Delta : C \otimes C' \xrightarrow[\Delta \otimes \Delta']{} C \otimes C \otimes C' \otimes C' \xrightarrow[id \otimes T_{C,C'} \otimes id]{} C \otimes C' \otimes C \otimes C' \qquad (\star)$$

Check if (☆) makes this commute:

$$\begin{tikzcd} D\otimes D\otimes D & D\otimes D \arrow[l, "\Delta\otimes id"'] \\ D\otimes D \arrow[u, "id\otimes\Delta"] & D \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

For $c\in C$ and $c'\in C'$

chase $c\otimes c'\in D$ around diag.

under either path, final image is

$$\sum_{(c)}\sum_{(c')} c_1\otimes c_1'\otimes c_2\otimes c_2'\otimes c_3\otimes c_3'$$

So diag commutes ✓

Also desire k-module hom

$$\varepsilon : C\otimes C' \to k$$

Guess

$$\varepsilon : C\otimes C' \xrightarrow{\varepsilon\otimes\varepsilon'} k\otimes k \xrightarrow{1\otimes 1\mapsto 1} k \qquad (☆☆)$$

Check (☆☆) makes this diag commute:

$$\begin{tikzcd} K \otimes D & D \otimes D \arrow[l, "\varepsilon \otimes id"'] \\ & D \arrow[u, "\Delta"'] \arrow[ul, leftrightarrow, "can"] \end{tikzcd}$$

For $c \in C$, $c' \in C'$ chase $c \otimes c' \in D$ around diag

$$\begin{tikzcd} \sum_{(c)}\sum_{(c')} \varepsilon(c_1)\,\varepsilon(c_1') \otimes c_2 \otimes c_2' & \sum_{(c)}\sum_{(c')} c_1 \otimes c_1' \otimes c_2 \otimes c_2' \arrow[l] \\ & c \otimes c' \arrow[u] \arrow[ul, leftrightarrow, "?"] \end{tikzcd}$$

Require

$$\sum_{(c)}\sum_{(c')} \varepsilon(c_1)\,\varepsilon(c_1')\, c_2 \otimes c_2' = c \otimes c'$$

$$\| $$

$$\sum_{(c)}\sum_{(c')} \varepsilon(c_1)\, c_2 \otimes \varepsilon(c_1')\, c_2'$$

$$\|$$

$$\underbrace{\left(\sum_{(c)} \varepsilon(c_1)\, c_2 \right)}_{= c} \otimes \underbrace{\left(\sum_{(c')} \varepsilon(c_1')\, c_2' \right)}_{= c'} \qquad \text{ok}$$

Similarly ($\star\star$) makes this diag commute

$$\begin{tikzcd} D \otimes D \arrow[r, "id \otimes \varepsilon"] & D \otimes K \arrow[dl, "can"] \\ D \arrow[u, "\Delta"] \arrow[ur] & \end{tikzcd}$$

We have turned $C \otimes C'$ into a coalg $\square$