Lec 19 Wednesday Oct 19

Recall fundamental involution

$$\omega : \Lambda \xrightarrow{S} \Lambda \xrightarrow{\sigma} \Lambda$$

where σ acts on each Λ_n as $(-1)^n id$

ω is Hopf alg iso and $\omega^2 = id$

Prop the map ω swaps

$$e_\lambda \leftrightarrow h_\lambda \qquad \lambda \in Par$$

pf ω sends

$$\begin{array}{ccccc} \Lambda & \xrightarrow{S} & \Lambda & \xrightarrow{\sigma} & \Lambda \\ e_\lambda & \to & (-1)^{|\lambda|} h_\lambda & \to & h_\lambda \\ h_\lambda & \to & (-1)^{|\lambda|} e_\lambda & \to & e_\lambda \end{array}$$

$\square$

Def For $\lambda \in Par$ define

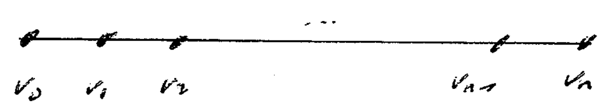
$$w_\lambda = \omega(m_\lambda)$$

Schur functions

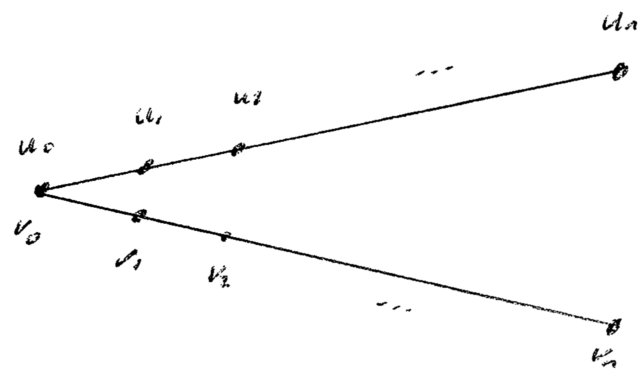
Motivation Let V denote a finite-free k-module

Given k-basis $v_0, v_1, \ldots, v_n$

Draw



Given a k-basis $u_0, u_1, \ldots, u_n$ for V



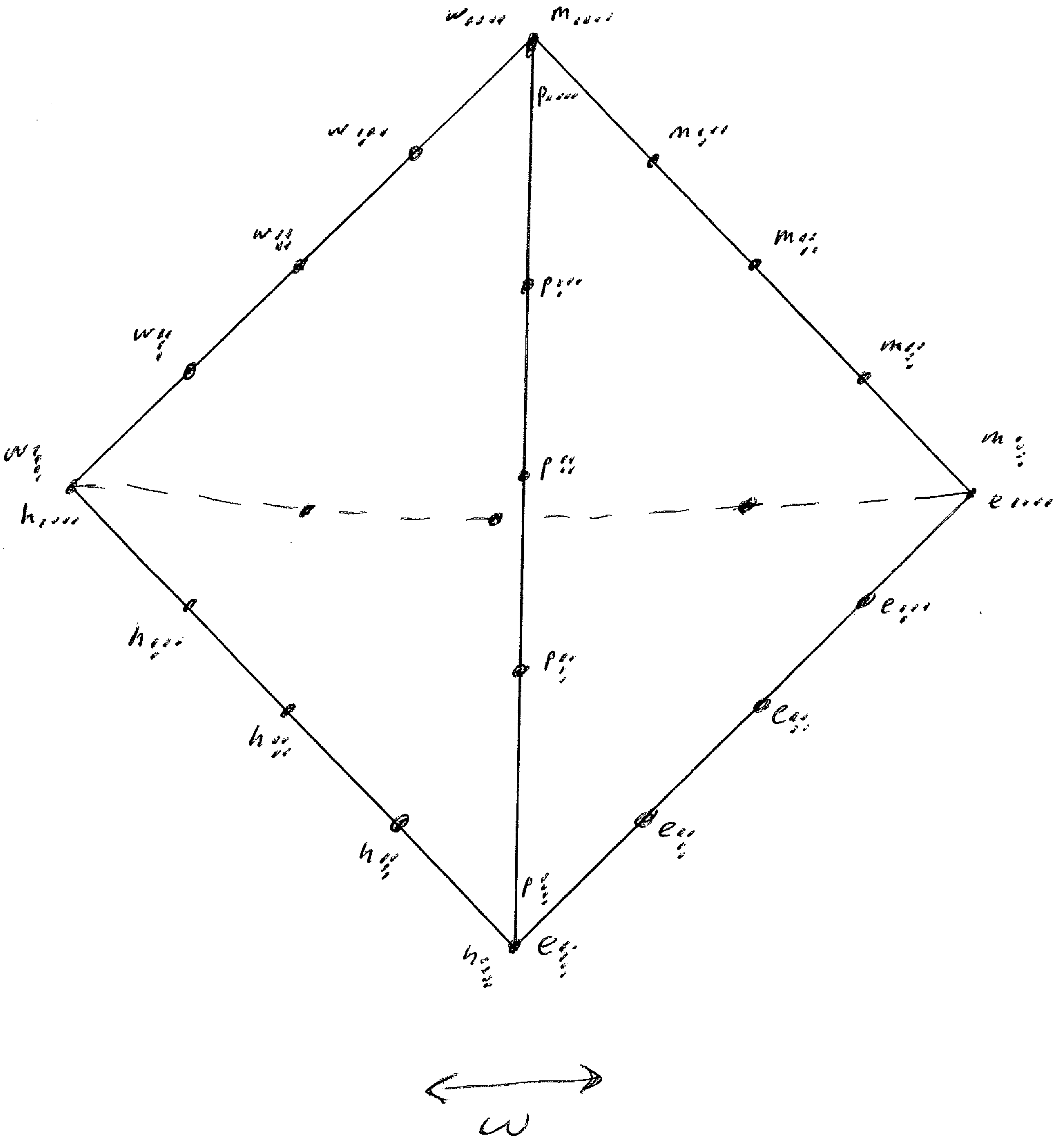
means

$$ku_0 + ku_1 + \cdots + ku_i = kv_0 + kv_1 + \cdots + kv_i \qquad 0 \le i \le n$$

"the transition matrices between $\{u_i\}_{i=0}^n$, $\{v_i\}_{i=0}^n$ are upper triangular and invertible"

Assume $\mathbb{Q}$ is subring of K 10/19/16

Describe our K-bases for Λ_n take $n=4$



What is K-basis along dashed line? Schur functions s_λ

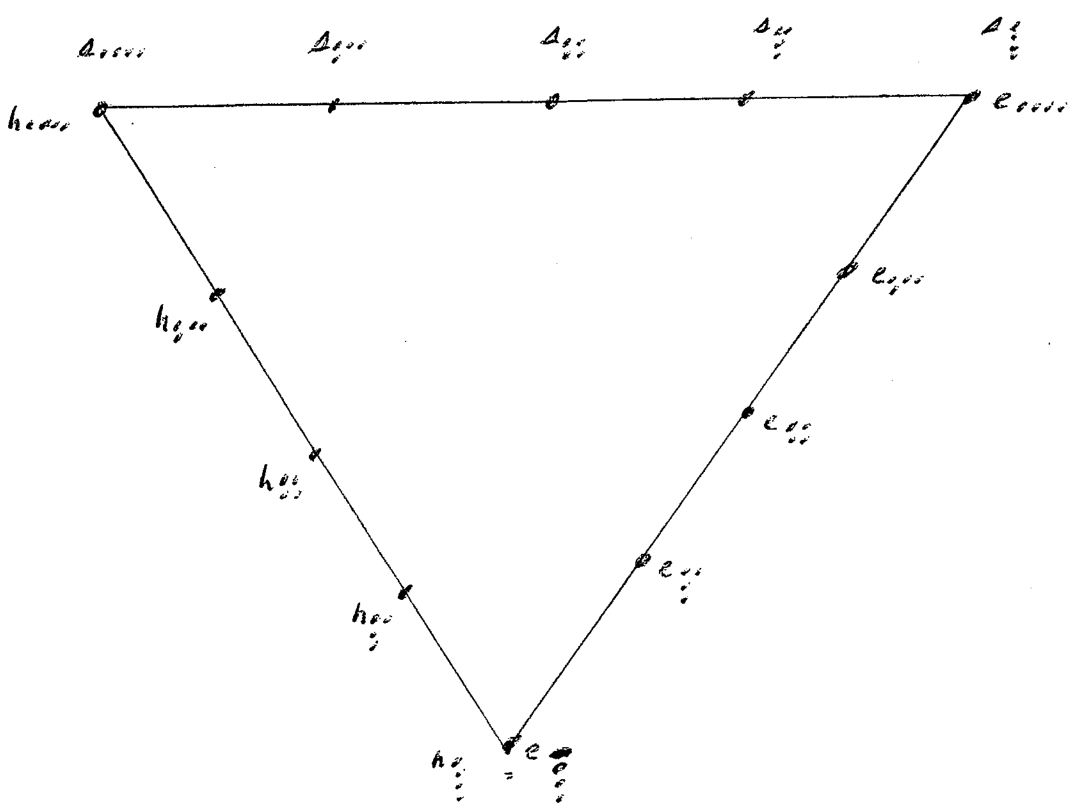
To define the Δ_λ we don't need the

$p_\lambda, m_\lambda, w_\lambda$

just e_λ, h_λ.

This allows us to drop the requirement that $\mathbb{Q}$ is a subring of K

For $n=4$:



As we will see, for $\lambda \in Par$

the Schur function s_λ is the unique element

in

$$\left(\sum_{\mu \trianglerighteq \lambda} K\, h_\mu \right) \cap \left(\sum_{\nu \trianglerighteq \lambda^t} K\, e_\nu \right)$$

that has h_λ-coef 1 (and e_{λ^t}-coef 1)

— o —

First we describe the e_λ, h_λ in more detail.

LEM the e_n, h_n are related as follows.

$$e_1 = h_1$$

$$e_2 = h_1^2 - h_2$$

$$e_3 = h_1^3 - 2h_1h_2 + h_3$$

$$e_4 = h_1^4 - 3h_1^2h_2 + h_2^2 + 2h_1h_3 - h_4$$

$$e_5 = h_1^5 - 4h_1^3h_2 + 3h_1h_2^2 + 3h_1^2h_3 - 2h_2h_3 - 2h_1h_4 + h_5$$

$$e_6 = h_1^6 - 5h_1^4h_2 + 6h_1^2h_2^2 - h_2^3 + 4h_1^3h_3 - 6h_1h_2h_3 - 3h_1^2h_4$$
$$+ h_3^2 + 2h_2h_4 + 2h_1h_5 - h_6$$

and

$$h_1 = e_1$$

$$h_2 = e_1^2 - e_2$$

$$h_3 = e_1^3 - 2e_1e_2 + e_3$$

$$h_4 = e_1^4 - 3e_1^2e_2 + e_2^2 + 2e_1e_3 - e_4$$

$$h_5 = e_1^5 - 4e_1^3e_2 + 3e_1e_2^2 + 3e_1^2e_3 - 2e_2e_3 - 2e_1e_4 + e_5$$

$$h_6 = e_1^6 - 5e_1^4e_2 + 6e_1^2e_2^2 - e_2^3 + 4e_1^3e_3 - 6e_1e_2e_3 - 3e_1^2e_4$$
$$+ e_3^2 + 2e_2e_4 + 2e_1e_5 - e_6$$

Pf use $e_nh_0 - e_{n-1}h_1 + e_{n-2}h_2 \cdots = 0$ for $n \geq 1$ $\square$

Obs the e_n, h_n are given by determinants

$$e_1 = |h_1| \qquad h_1 = |e_1|$$

$$e_2 = \begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix} \qquad h_2 = \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix}$$

$$e_3 = \begin{vmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} \qquad h_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix}$$

$$e_4 = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix} \qquad h_4 = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & e_1 & e_2 & e_3 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{vmatrix}$$

$$e_5 = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 & h_3 & h_4 \\ 0 & 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 & h_2 \\ 0 & 0 & 0 & 1 & h_1 \end{vmatrix} \qquad h_5 = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & e_1 & e_2 & e_3 & e_4 \\ 0 & 1 & e_1 & e_2 & e_3 \\ 0 & 0 & 1 & e_1 & e_2 \\ 0 & 0 & 0 & 1 & e_1 \end{vmatrix}$$

$\cdots \qquad \cdots$

LEM For $n \geq 1$, e_n and h_n are given by determinants

$$e_n = \begin{vmatrix} h_1 & h_2 & \cdots & h_n \\ 1 & h_1 & h_2 \cdots & \vdots \\ & 1 & h_1 & \ddots \\ & & 1 \ddots & h_2 \\ 0 & & 1 & h_1 \end{vmatrix} \qquad h_n = \begin{vmatrix} e_1 & e_2 & \cdots & e_n \\ 1 & e_1 & e_2 & \vdots \\ & 1 & e_1 & \ddots \\ & & \ddots & e_2 \\ 0 & & 1 & e_1 \end{vmatrix}$$

$$\parallel$$

$$E_n$$

pf show $e_n = E_n$ for $n \geq 1$

Expanding the det E_n along top row, get

$$0 = \sum_{i=0}^{n} (-1)^i E_i h_{n-i} \qquad (E_0 = 1)$$

We saw earlier

$$0 = \sum_{i=0}^{n} (-1)^i e_i h_{n-i} \qquad (e_0 = 1)$$

e_n, E_n satisfy the same rec and init conds, so $e_n = E_n$ $\square$

Motivation, cont

Find s_λ for $\lambda \in \mathrm{Par}_n$, $n=4$

K-bases for Λ_4:

$e_1^4 \quad e_1^2 e_2 \quad e_2^2 \quad e_1 e_3 \quad e_4$ *

$h_1^4 \quad h_1^2 h_2 \quad h_2^2 \quad h_1 h_3 \quad h_4$ **

Write ** in terms of *

	h_1^4	$h_1^2 h_2$	h_2^2	$h_1 h_3$	h_4
e_1^4	1	1	1	1	1
$e_1^2 e_2$		-1	-2	-2	-3
e_2^2			1	0	1
$e_1 e_3$				1	2
e_4					-1

For $0 \le i \le 4$ find a K-basis for

$$\left(K\text{-span of last } 5-i \text{ terms in } *\right) \cap \left(K\text{-span of last } i+1 \text{ terms in } **\right)$$

i	K-basis sol	name
0	$h_4 = e_1^4 - 3e_1^2e_2 + e_2^2 + 2e_1e_3 - e_4$	$s_{(4)}$
1	$h_1h_3 - h_4 = e_1^2e_2 - e_2^2 - e_1e_3 + e_4$	$s_{(3,1)}$
2	$h_2^2 - h_1h_3 = e_2^2 - e_1e_3$	$s_{(2,2)}$
3	$h_1^2h_2 - h_2^2 - h_1h_3 + h_4 = e_1e_3 - e_4$	$s_{(2,1,1)}$
4	$h_1^4 - 3h_1^2h_2 + h_2^2 + 2h_1h_3 - h_4 = e_4$	$s_{(1,1,1,1)}$

the common values are given by dets

$$\Delta_{1111} = \left| h_4 \right| = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & e_1 & e_2 & e_3 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta_{\substack{111\\1}} = \begin{vmatrix} h_3 & h_4 \\ 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_2 & e_3 & e_4 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta_{\substack{11\\11}} = \begin{vmatrix} h_2 & h_3 \\ h_1 & h_2 \end{vmatrix} = \begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix}$$

$$\Delta_{\substack{11\\1\\1}} = \begin{vmatrix} h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_3 & e_4 \\ 1 & e_1 \end{vmatrix}$$

$$\Delta_{\substack{1\\1\\1\\1}} = \begin{vmatrix} h_1 & h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix} = \left| e_4 \right|$$

The pattern suggests:

For $\lambda \in Par$,

$$\Delta_\lambda = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \cdots \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \cdots \\ & & \ddots & \ddots \\ & \cdots & h_{\lambda_\ell-1} & h_{\lambda_\ell} \end{vmatrix} \qquad \ell = \text{length}(\lambda)$$

$$= \det\left(h_{\lambda_i+j-i}\right)_{1\le i,j\le \ell}$$

$$\left[h_0 = 1, \quad h_j = 0 \text{ if } j<0\right]$$

$$\Delta_\lambda = \begin{vmatrix} e_{\mu_1} & e_{\mu_1+1} & e_{\mu_1+2} & \cdots \\ e_{\mu_2-1} & e_{\mu_2} & e_{\mu_2+1} & \cdots \\ \vdots & & \ddots & \\ & & & \vdots \\ & \cdots & e_{\mu_k-1} & e_{\mu_k} \end{vmatrix} \qquad \begin{matrix} \mu = \lambda^t \\ k = \text{length}(\mu) \end{matrix}$$

$$= \det\left(e_{\mu_i+j-i}\right)_{1\le i,j\le k}$$

$$\left[e_0 = 1, \quad e_j = 0 \text{ if } j<0\right]$$

Another example

$n=5$ Compare

$e_1^5, \quad e_1^3 e_2, \quad e_1 e_2^2, \quad e_1^2 e_3, \quad e_2 e_3, \quad e_1 e_4, \quad e_5$ $\quad *$

$h_1^5, \quad h_1^3 h_2 \quad h_1 h_2^2 \quad h_1^2 h_3 \quad h_2 h_3 \quad h_1 h_4 \quad h_5$ $\quad **$

Express $**$ in terms of $*$

	h_1^5	$h_1^3 h_2$	$h_1 h_2^2$	$h_1^2 h_3$	$h_2 h_3$	$h_1 h_4$	h_5
e_1^5	1	1	1	1	1	1	1
$e_1^3 e_2$		-1	-2	-2	-3	-3	-4
$e_1 e_2^2$			1	0	2	1	3
$e_1^2 e_3$				1	1	2	3
$e_2 e_3$					-1	0	-2
$e_1 e_4$						-1	-2
e_5							1

For $0 \le i \le 6$ find K-basis for

(K-span of last $7-i$ terms in *) $\cap$ (K-span of last ith terms in **)

i	K-basis set	name
0	$h_5 = e_1^5 - 4e_1^3e_2 + 3e_1e_2^2 + 3e_1^2e_3 - 2e_2e_3 - 2e_1e_4 + e_5$	S
1	$h_1h_4 - h_5 = e_1^3e_2 - 2e_1e_2^2 - e_1^2e_3 + 2e_2e_3 + e_1e_4 - e_5$	S
2	$h_2h_3 - h_1h_4 = e_1e_2^2 - e_1^2e_3 - e_2e_3 + e_1e_4$	S
3	$h_1^2h_3 - h_2h_3 - h_1h_4 + h_5 = e_1^2e_3 - e_2e_3 - e_1e_4 + e_5$	S
4	$h_1h_2^2 - h_1^2h_3 - h_2h_3 + h_1h_4 = e_2e_3 - e_1e_4$	S
5	$h_1^3h_2 - 2h_1h_2^2 - h_1^2h_3 + 2h_2h_3 + h_1h_4 - h_5 = e_1e_4 - e_5$	S
6	$h_1^5 - 4h_1^3h_2 + 3h_1h_2^2 + 3h_1^2h_3 - 2h_2h_3 - 2h_1h_4 + h_5 = e_5$	S

$$\Delta = \left|h_5\right| = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 1 & e_1 & e_2 & e_3 & e_4 \\ 0 & 1 & e_1 & e_2 & e_3 \\ 0 & 0 & 1 & e_1 & e_2 \\ 0 & 0 & 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} h_4 & h_5 \\ 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_2 & e_3 & e_4 & e_5 \\ 1 & e_1 & e_2 & e_3 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} h_3 & h_4 \\ h_1 & h_2 \end{vmatrix} = \begin{vmatrix} e_2 & e_3 & e_4 \\ e_1 & e_2 & e_3 \\ 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_3 & e_4 & e_5 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_3 & e_4 \\ e_1 & e_2 \end{vmatrix}$$

$$\Delta \begin{vmatrix} h_2 & h_3 & h_4 & h_5 \\ 1 & h_1 & h_2 & h_3 \\ 0 & 1 & h_1 & h_2 \\ 0 & 0 & 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_4 & e_5 \\ 1 & e_1 \end{vmatrix}$$

$n = 3$

$e_1^3 \qquad e_1 e_2 \qquad e_3$ $\qquad *$

$h_1^3 \qquad h_1 h_2 \qquad h_3$ $\qquad **$

	h_1^3	$h_1 h_2$	h_3
e_1^3	1	1	1
$e_1 e_2$		-1	-2
e_3			1

i		
0	$h_3 = e_1^3 - 2 e_1 e_2 + e_3$	$\Delta \dots$
1	$h_1 h_2 - h_3 = e_1 e_2 - e_3$	Δ
2	$h_1^3 - 2 h_1 h_2 + h_3 = e_3$	Δ

$n=3$

$$\Delta_{\square\square\square} = \left| h_3 \right| = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix}$$

$$\Delta_{\substack{\square\square \\ \square}} = \begin{vmatrix} h_2 & h_3 \\ 1 & h_1 \end{vmatrix} = \begin{vmatrix} e_2 & e_3 \\ 1 & e_1 \end{vmatrix}$$

$$\Delta_{\substack{\square \\ \square \\ \square}} = \begin{vmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} = \left| e_3 \right|$$

$n = 1, 2$

$n = 1$

$$h_1 = e_1 \qquad |h_1| = |e_1| = s_1$$

$n = 2$

$$h_2 = e_1^2 - e_2 \qquad |h_2| = \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix} = s_{11}$$

$$h_1^2 - h_2 = e_2 \qquad \begin{vmatrix} h_1 & h_2 \\ 1 & h_1 \end{vmatrix} = |e_2| = s_{11}$$