Lec 18 Monday Oct 17

Recall: Fa $n \in \mathbb{N}$ and $\lambda \in Par_n$

write

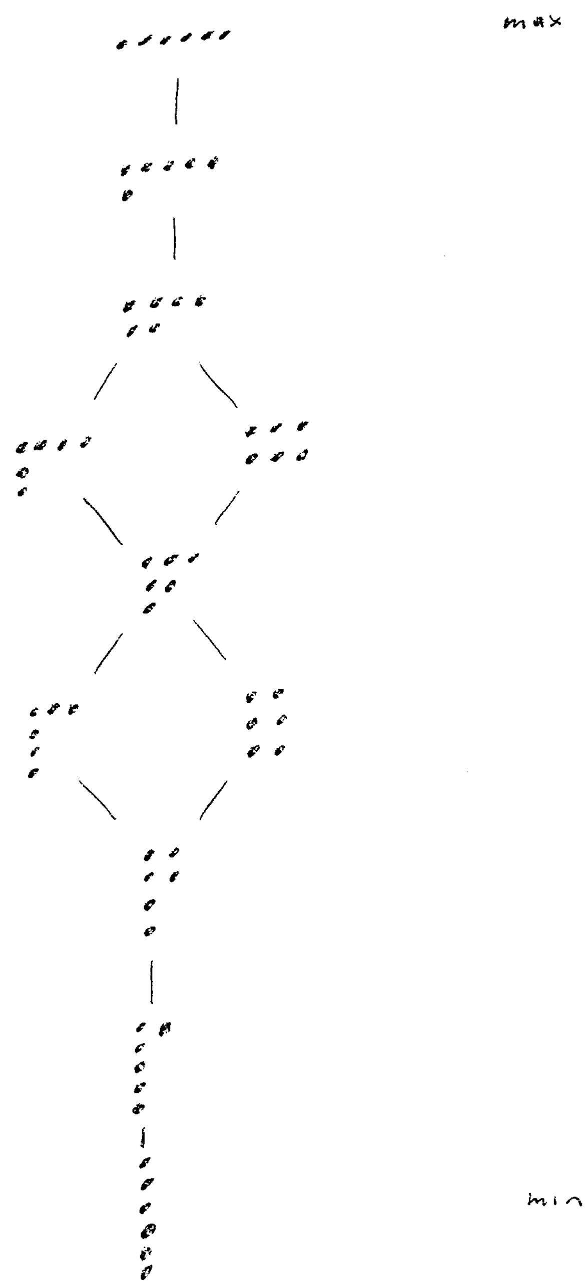
$$e_\lambda = \sum_{\mu \in Par_n} a_\mu^\lambda m_\mu$$

a_μ^λ = # of (0,1)-matrices that have col sums $\mu_1, \mu_2, \dots$ and row sums $\lambda_1, \lambda_2, \vdots$

Next consider when is $a_\mu^\lambda = 0$

For $n \in \mathbb{N}$ we now consider a partial order $\trianglelefteq$ on P_n called dominance order

Ex $n=6$ Hasse diagram for $\trianglelefteq$ is



Ex $n=6$, cont

$\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5\ \lambda_6$

$$\begin{tikzcd}[column sep=small]
 & 600000 \arrow[d, dash] & \\
 & 510000 \arrow[d, dash] & \\
 & 420000 \arrow[dl, dash] \arrow[dr, dash] & \\
411000 \arrow[dr, dash] & & 330000 \arrow[dl, dash] \\
 & 321000 \arrow[dl, dash] \arrow[dr, dash] & \\
311100 \arrow[dr, dash] & & 222000 \arrow[dl, dash] \\
 & 221100 \arrow[d, dash] & \\
 & 211110 \arrow[d, dash] & \\
 & 111111 &
\end{tikzcd}$$

partial sums

$\lambda_1,\ \lambda_1+\lambda_2,\ \lambda_1+\lambda_2+\lambda_3,\ \ldots\ \lambda_1+\cdots+\lambda_6$

$$\begin{tikzcd}[column sep=small]
 & 666666 \arrow[d, dash] & \\
 & 566666 \arrow[d, dash] & \\
 & 466666 \arrow[dl, dash] \arrow[dr, dash] & \\
456666 \arrow[dr, dash] & & 366666 \arrow[dl, dash] \\
 & 356666 \arrow[dl, dash] \arrow[dr, dash] & \\
345666 \arrow[dr, dash] & & 246666 \arrow[dl, dash] \\
 & 245666 \arrow[d, dash] & \\
 & 234566 \arrow[d, dash] & \\
 & 123456 &
\end{tikzcd}$$

Def For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$

define

$$\lambda \trianglelefteq \mu \text{ whenever } \lambda_1 + \lambda_2 + \cdots + \lambda_k \le \mu_1 + \mu_2 + \cdots + \mu_k$$

$$\text{for } k = 1, 2, 3 \ldots$$

Write

$$\lambda \vartriangleleft \mu \text{ whenever } \lambda \trianglelefteq \mu \text{ and } \lambda \ne \mu$$

Def For $\lambda \in \mathrm{Par}$

The <u>transpose</u> partition λ^t is obtained from λ by reflecting the Ferrar diagram rows $\leftrightarrow$ cols

ex $\lambda = \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{matrix}$ $\qquad \lambda^t = \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \\ \bullet & \end{matrix}$

LEM For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$,

$$\lambda \trianglelefteq \mu \text{ iff } \mu^t \trianglelefteq \lambda^t$$

pf (ex)

$\square$

LEM Fn $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$

(i) $a^{\lambda}_{\mu} = 1$ if $\lambda = \mu^t$

(ii) $a^{\lambda}_{\mu} = 0$ if $\lambda \ntrianglelefteq \mu^t$

pf (i) ex

(ii) We assume $a^{\lambda}_{\mu} \neq 0$ and show $\lambda \trianglelefteq \mu^t$

$\exists$ (0,1)-matrix with row/col sums

$$\begin{array}{c|c} & \mu_1 \ \mu_2 \cdots \\ \hline \lambda_1 & \\ \lambda_2 & \\ \vdots & \end{array}$$

Fn $k = 1, 2, ..$ show

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \leq \mu_1^t + \mu_2^t + \cdots + \mu_k^t$$

note that

μ_i^t = # parts among $\mu_1, \mu_2, \ldots$ that are at least i

$\lambda_1 + \lambda_2 + \cdots + \lambda_k$ = # 1's in rows $1, 2, \ldots, k$

= k (# cols that have k 1's in rows $1, 2, \ldots, k$)

+ $(k-1)$ (... $k-1$ ...)

+ ...

+ 1 (... 1 ...)

= 1 (# cols that have k 1's in rows $1, 2, \ldots, k$)

+ 1 (... $k-1$ or k ...)

+ 1 (... $k-2$ or $k-1$ or k ...)

+ ...

+ (... 1 or 2 or ... or k ...)

$\leq$ # cols that have at least k 1's $(= u_k^t)$

+ ------ $k-1$ $(= u_{k-1}^t)$

+ ------ $k-2$ $(= u_{k-2}^t)$

$\vdots$

+ ... 1 $(= u_1^t)$

+ ...

$\leq u_1^t + u_2^t + \cdots + u_k^t$ $\square$

COR For $n \in \mathbb{N}$ the following is a k-basis for Λ_n:

$$e_\lambda \qquad \lambda \in Par_n \qquad *$$

pf Write the e_λ in the basis

$$m_\lambda \qquad \lambda \in Par_n. \qquad **$$

With respect to an approp ordering of $*$, $**$ the coef matrix is upper triangular with all diag entries 1. Hence this coef matrix is invertible.

Result follows. $\square$

Cor The elements

$$e_n \qquad n = 1, 2, 3, \dots$$

are algebraically indep over K, and generate Λ

In other words $\exists$ K-alg iso

$$\Lambda \to K[x_1, x_2, \dots]$$

$\leftarrow$ poly alg

that sends

$$e_i \to x_i \quad \text{for } i \geq 1.$$

pf By prev Cor and def of the e_n $\square$

LEM For $n \in \mathbb{N}$ the following is a k-basis for Λ_n

$$h_\lambda \qquad \lambda \in \mathrm{Par}_n$$

pf Recall

$$e_\lambda \qquad \lambda \in \mathrm{Par}_n \qquad *$$

is a k-basis for Λ_n

Apply S to $*$

Recall $S : \Lambda_n \to \Lambda_n$ is k-module hom

$S^2 = id$ so S is bij

Also $S(e_\lambda) = (-1)^n h_\lambda \qquad \forall \lambda \in \mathrm{Par}_n$

Result follows. $\square$

Cor The elements

h_n $n = 1, 2, 3 \ldots$

are alg indep and generate Λ.

pf By prev lem and def of h_λ $\square$

For $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$ write

$$p_\lambda = \sum_{\mu \in \mathrm{Par}_n} b^\lambda_\mu \, m_\mu \qquad b^\lambda_\mu \in K$$

Find b^λ_μ

LEM For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$

b^λ_μ = # ways to partition the set

$\{1, 2, \ldots, \ell\}$ $\qquad \ell = \text{length}(\lambda)$

into subsets

$S_1, S_2, \ldots, S_k$ $\qquad k = \text{length}(\mu)$

such that

$$\mu_i = \sum_{j \in S_i} \lambda_j \qquad 1 \le i \le k$$

pf b^λ_μ = coef of m_μ in p_λ

= coef of $x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ in

$$\left(\sum_{i=1}^\infty x_i^{\lambda_1}\right)\left(\sum_{i=1}^\infty x_i^{\lambda_2}\right) \cdots \left(\sum_{i=1}^\infty x_i^{\lambda_\ell}\right)$$

Result follows. $\square$

LEM For $n \in \mathbb{N}$ and $\lambda, \mu \in Par_n$

(i) $b^\lambda_\mu =$

$|\{i \mid \lambda_i = 1\}|! \; |\{i \mid \lambda_i = 2\}|! \; \cdots$ if $\lambda = \mu$

(ii) $b^\lambda_\mu = 0$ if $\lambda \ntrianglerighteq \mu$.

$\square$

Pf Routine

Cor Assume $\mathbb{Q}$ is a subring of K

then for $n \in \mathbb{N}$ the following is a K-basis

for Λ_n:

$$p_\lambda \qquad \lambda \in Par_n \qquad *$$

pf Write the p_λ in the basis

$$m_\lambda \qquad \lambda \in Par_n \qquad **$$

With resp to an approp ordering of $*, **$

the coef matrix is upper tr with diag entries pos integers $\subseteq \mathbb{Q}^{\subseteq}$.

Hence coef matrix is invertible.

Result follows. $\square$

Cor Assume $\mathbb{Q}$ is subring of K

then the elements

$p_n \qquad n = 1, 2, 3 \ldots$

are alg indep and generate Λ

pf By prev cor and def of p_λ $\square$

LEM The antipode $S : \Lambda \to \Lambda$ is an iso of Hopf algebras.

pf Since Λ is commutative and cocommutative. $\square$

The fundamental involution

Define a K-module hom

$$\sigma : \Lambda \to \Lambda$$

such that for $n \in \mathbb{N}$,

σ acts on Λ_n as $(-1)^n id$

Obs σ is K-alg iso and $\sigma^2 = id$

Also σ, S commute.

LEM $\sigma : \Lambda \to \Lambda$ is an iso of Hopf algebras.

pf Show σ respects Δ

$\forall \lambda \in Par$

$$\Delta(m_\lambda) = \sum_{\substack{\mu,\nu \in Par \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$$

Require

$$\begin{array}{ccccc} \Delta(\sigma(m_\lambda)) & \stackrel{?}{=} & \sum_{\mu,\nu} & \sigma(m_\mu) & \otimes\ \sigma(m_\nu) \\ \| & & & \| & \| \\ (-1)^{|\lambda|} m_\lambda & & & (-1)^{|\mu|} m_\mu & (-1)^{|\nu|} m_\nu \end{array}$$

$|\mu| + |\nu| = |\lambda|$

OK

Show σ respects ε:

$\forall \lambda \in Par$ require

$$\varepsilon(\sigma(m_\lambda)) \overset{?}{=} \varepsilon(m_\lambda) \qquad \text{ok}$$

Each side is $\begin{cases} 1 & \text{if } \lambda = \emptyset \\ 0 & \text{if } \lambda \neq \emptyset \end{cases}$

$\square$

<u>Def</u> Define $\omega : \Lambda \to \Lambda$ to be the composition

$$\omega : \Lambda \xrightarrow[S]{} \Lambda \xrightarrow[\sigma]{} \Lambda$$

Obs ω is an iso of Hopf algebras and

$$\omega^2 = id$$

Call ω the <u>fundamental involution</u>