Lec 17 Friday Oct 14 10/14/16

1

LEM The map $\Delta : \Lambda \to \Lambda \otimes \Lambda$

is coassoc

pf Since for $\lambda, \mu, \nu \in Par$

$$(\lambda \sqcup \mu) \sqcup \nu = \lambda \sqcup (\mu \sqcup \nu)$$

$\square$

We now define the counit

$$\varepsilon : \Lambda \to K.$$

Define a K-module hom $\varepsilon : \Lambda \to K$ st

$$\varepsilon(\Lambda_n) = 0 \quad \text{for } n \geq 1$$

$$\varepsilon(1) = 1.$$

Obs ε is K-alg hom.

Prop. Above Δ, ε turn Λ into a bialgebra

pf Δ, ε are K-alg morphisms.

check diagrams

$$\begin{tikzcd}
K \otimes \Lambda & \Lambda \otimes \Lambda \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & \Lambda \otimes K \\
 & \Lambda \arrow[ul, "can"] \arrow[u, "\Delta"'] \arrow[ur, "can"'] \arrow[ul, phantom, "?"] \arrow[ur, phantom, "?"] &
\end{tikzcd}$$

0 unless $\mu = \emptyset$

$$\begin{tikzcd}
\underbrace{\sum_{\mu \cup \nu = \lambda} \varepsilon(m_\mu) \otimes m_\nu}_{=\, 1 \otimes m_\lambda} & \sum_{\mu \cup \nu = \lambda} m_\mu \otimes m_\nu \arrow[l] \\
 & m_\lambda \arrow[u] \arrow[ul, "ok"]
\end{tikzcd}$$

$\square$

By constr bialg Λ is graded, connected, commutative, cocommutative.

So Λ has antipode S and becomes Hopf alg

LEM In the Hopf alg Λ, the power sym function p_n is primitive for $n \geq 1$.

pf Show

$$\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$$

Recall $p_n = m_\lambda$ for $\lambda = (n)$

$$\Delta(m_\lambda) = \sum_{\substack{\mu, \nu \in \mathrm{Par} \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$$

Only sols for μ, ν are

- $\mu = \phi, \quad \nu = \lambda$
- $\mu = \lambda, \quad \nu = \phi$

Result follows. $\square$

LEM Fa $n \in \mathbb{N}$,

(i) $$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}$$

(ii) $$\Delta(h_n) = \sum_{i=0}^{n} h_i \otimes h_{n-i}$$

pf (i) This

$$\Delta(m_\lambda) = \sum_{\substack{\mu,\nu \in Par \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$$

fa $\lambda = (\underbrace{1,1,\ldots,1}_{n})$

(ii) $$\Delta(h_n) = \Delta\left(\sum_{\lambda \in Par_n} m_\lambda\right)$$

$$= \sum_{\lambda \in Par_n} \sum_{\substack{\mu,\nu \in Par \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$$

$$= \sum_{i=0}^{n} \sum_{\mu \in Par_i} \sum_{\nu \in Par_{n-i}} m_\mu \otimes m_\nu$$

$$= \sum_{i=0}^{n} \underbrace{\left(\sum_{\mu \in Par_i} m_\mu\right)}_{= h_i} \otimes \underbrace{\left(\sum_{\nu \in Par_{n-i}} m_\nu\right)}_{= h_{n-i}}$$

$\square$

Next goal: Describe S action on

p_n, e_n, h_n

Recall $S^2 = id$ since Λ is com and cocom.

LEM For $n \geq 1$

$$S(p_n) = -p_n$$

pf Since p_n is primitive $\square$

For an indet t consider the generating function

$$G(t) = (1+x_1 t)(1+x_2 t) \cdots$$
$$= \prod_{i=1}^{\infty} (1 + x_i t)$$

LEM We have

(i) $G(t) = \sum_{n \in \mathbb{N}} e_n t^n$

(ii) $(G(t))^{-1} = \sum_{n \in \mathbb{N}} (-1)^n h_n t^n$

(iii) For $n \geq 1$,

$$0 = \sum_{i=0}^{n} e_i h_{n-i} (-1)^i$$

pf (i) For $n \in \mathbb{N}$ the coef of t^n in $G(t)$ is

$$\underbrace{\sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}}_{= e_n}$$

(ii) $$(G(-t))^{-1} = \prod_{i=1}^{\infty} \frac{1}{1-x_i t}$$

$$= \prod_{i=1}^{\infty} \left(1 + x_i t + x_i^2 t^2 + \cdots\right)$$

Expanding this, for $n \in \mathbb{N}$ the coef of t^n is

$$\sum_{\substack{a_1, a_2, \ldots \in \mathbb{N} \\ n = a_1 + a_2 + \cdots}} x_1^{a_1} x_2^{a_2} \cdots$$

$$= \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= h_n.$$

Result follows.

(iii) By (i), (ii)

$$1 = \left(\sum_{i \in \mathbb{N}} e_i t^i\right)\left(\sum_{j \in \mathbb{N}} (-1)^j h_j t^j\right)$$

For $n \ge 1$,

$$0 = \text{coef of } t^n \text{ in RHS}$$

$$= \sum_{i=0}^{n} e_i h_{n-i} (-1)^{n-i}$$

Result follows.

$\square$

LEM For $n \in \mathbb{N}$

(i) $S(e_n) = (-1)^n h_n$

(ii) $S(h_n) = (-1)^n e_n$

pf (i) By induction on n

$n = 0$ ✓ $\quad e_0 = 1 = h_0$ ✓

$n \geq 1$:

$\forall a \in \Lambda$

$$\varepsilon(a) 1_\Lambda = \sum_{(a)} a_1 S(a_2)$$

take $a = e_n$

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}$$

$$\varepsilon(e_n) = 0$$

so

$$0 = \sum_{i=0}^{n} e_i S(e_{n-i})$$

$$= \underset{\overset{\shortparallel}{1}}{e_0} S(e_n) + \sum_{i=1}^{n} e_i \underbrace{S(e_{n-i})}_{\substack{\shortparallel \text{ by ind} \\ (-1)^{n-i} h_{n-i}}}$$

$$S(e_n) = -(-1)^n \sum_{i=1}^{n} e_i h_{n-i} (-1)^i$$

$$= (-1)^n h_n \qquad \text{by prev LEM}$$

(ii) By (i) and $S^2 = id$ $\square$

<u>Notation</u> Fn $\lambda \in Par$

write

$$\lambda = (\lambda_1, \lambda_2, \dots \lambda_\ell)$$

Define

$$e_\lambda = \prod_{i=1}^{\ell} e_{\lambda_i}$$

$$p_\lambda = \prod_{i=1}^{\ell} p_{\lambda_i}$$

$$h_\lambda = \prod_{i=1}^{\ell} h_{\lambda_i}$$

Each of

e_λ, p_λ, h_λ

has degree $|\lambda|$

For instance

$$\deg(e_\lambda) = \sum_{i=1}^{\ell} \deg(e_{\lambda_i}) = \sum_{i=1}^{\ell} \lambda_i = |\lambda|$$

LEM For $\lambda \in Par$

(i) $S(p_\lambda) = (-1)^{\ell} p_\lambda$ $\ell =$ length of λ

(ii) $S(e_\lambda) = (-1)^{|\lambda|} h_\lambda$

(iii) $S(h_\lambda) = (-1)^{|\lambda|} e_\lambda$

pf Write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

(i) $$S(p_\lambda) = S\left(\prod_{i=1}^{\ell} p_{\lambda_i}\right)$$
$$= \prod_{i=1}^{\ell} S(p_{\lambda_i})$$
$$= \prod_{i=1}^{\ell} -p_{\lambda_i}$$
$$= (-1)^{\ell} p_\lambda$$

(ii) $$S(e_\lambda) = S\left(\prod_{i=1}^{\ell} e_{\lambda_i}\right)$$
$$= \prod_{i=1}^{\ell} S(e_{\lambda_i})$$
$$= \prod_{i=1}^{\ell} (-1)^{\lambda_i} h_{\lambda_i}$$
$$= (-1)^{|\lambda|} h_\lambda$$

(iii) Sim to (ii) $\square$

Recall that for $n \in \mathbb{N}$,

$$m_\lambda \qquad \lambda \in \mathrm{Par}_n \qquad (*)$$

is a K-basis for the finite-free K-module Λ_n

Next goal: express

$$e_\lambda \qquad \lambda \in \mathrm{Par}_n$$

in the basis $(*)$

Ex $n=3$ One checks the coef matrix is

	$e_{(3)}$	$e_{(2,1)}$	$e_{(1,1,1)}$
$m_{(1,1,1)}$	1	3	6
$m_{(2,1)}$	0	1	3
$m_{(3)}$	0	0	1

We recall

$$m_{\cdot\cdot\cdot} = [x_1^3]$$

$$m_{\substack{\cdot\,\cdot\\ \cdot}} = [x_1^2 x_2]$$

$$m_{\substack{\cdot\\ \cdot\\ \cdot}} = [x_1 x_2 x_3]$$

$$e_{\cdot\cdot\cdot} = e_3 \qquad e_1 = [x_1]$$

$$e_{\substack{\cdot\,\cdot\\ \cdot}} = e_2 e_1 \qquad e_2 = [x_1 x_2]$$

$$e_{\substack{\cdot\\ \cdot\\ \cdot}} = e_1^3 \qquad e_3 = [x_1 x_2 x_3]$$

We ask what do above coefs count?

For $\lambda \in Par_n$ write

$$e_\lambda = \sum_{\mu \in Par_n} a_\mu^\lambda \, m_\mu$$

Fr $\mu \in Par_n$

a^{λ}_{μ} = # ways to write the monomial

$$x_1^{\mu_1} x_2^{\mu_2} \cdots$$

as a product of:

- a summand in $[x_1 x_2 \cdots x_{\lambda_1}]$
- a summand in $[x_1 x_2 \cdots x_{\lambda_2}]$

$\cdots$

= # ways to put the multiset

$$\{\mu_1 \cdot x_1, \mu_2 \cdot x_2, \ldots\}$$

into

box$_1$ of size λ_1

box2 of size λ_2

$\cdots$

st each box contains no duplicates

Represent each sol by a (0,1)-matrix

	x_1 supply	x_2 supply	$\cdots$
box 1			
box 2			
$\vdots$			

(i,j)-entry is $\begin{cases} 1 & \text{if box } i \text{ gets } x_j \\ 0 & \text{else} \end{cases}$

the sols corresp to the (0,1)-matrices with

col sums

$\mu_1 \quad \mu_2 \quad \cdots$

and row sums $\quad \lambda_1$

λ_2

$\vdots$

We have shown..

LEM For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$,

a^λ_μ = # (0,1)-matrices [with rows/cols indexed by 1, 2, ...] that have column sums

$\mu_1, \mu_2, \ldots$

and row sums $\begin{matrix}\lambda_1\\ \lambda_2\\ \vdots\end{matrix}$

$\square$

Cor For $n \in \mathbb{N}$ and $\lambda, \mu \in \mathrm{Par}_n$,

$$a^\lambda_\mu = a^\mu_\lambda.$$

pf By prev Cor. $\square$