II Symmetric functions

Fix commutative ring K as before.

Given mutually commuting indeterminates $x_1, x_2, \ldots$

A <u>weak composition</u> is a sequence

parts of α

$$\alpha = (\alpha_1, \alpha_2, \ldots) \qquad \alpha_i \in \mathbb{N}$$

fin many α_i nonzero

The corresp <u>monomial</u> is

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

Define

$$\text{degree}(x^\alpha) = \alpha_1 + \alpha_2 + \cdots$$

Given an ∞ K-linear combination of monomials

$$f = \sum_{\substack{\alpha \\ \alpha \text{ a weak comp}}} c_\alpha x^\alpha \qquad c_\alpha \in K$$

We say f has <u>bounded degree</u> whenever

$\exists\ N \in \mathbb{N}$ st

$$\deg(x^\alpha) > N \quad \text{implies} \quad c_\alpha = 0$$

For example

$$x_1^2 + x_2^2 + x_3^2 + \cdots$$

has bounded degree but

$$1 + x_1 + x_1^2 + x_1^3 + \cdots$$

does not.

Define

$R(x)$ = set of all ∞ K-linear combinations of monomials that have bounded degree.

$R(x)$ is K-algebra under usual mult and unit.

Algebra $R(x)$ is graded by degree:

$$R(x) = \sum_{n \in \mathbb{N}} R(x)_n$$

The symmetric group S_∞ acts on $R(x)$ by permuting

$x_1, x_2, \ldots$

S_∞ leaves $R(x)_n$ invariant $\forall n \in \mathbb{N}$

Define a subgroup

$$S_{(\infty)} = \{ \sigma \in S_\infty \mid \sigma \text{ leaves fixed all but finitely many } x_i \}$$

$\forall\, v \in R(x)$ let

$S_{(\infty)} v$ = orbit of $S_{(\infty)}$ that contains v

$$[v] = \sum_{w \in S_{(\infty)} v} w$$

Obs $\quad [v] \in R(x)$

Define

$$\Lambda(x) = \{ f \in R(x) \mid f \text{ is fixed by all } \sigma \in S_{(\infty)} \}$$

" "symmetric functions"

Λ

Obs

Λ is K- subalgebra of $R(x)$

Λ inherits a grading

$$\Lambda = \sum_{n \in \mathbb{N}} \Lambda_n \qquad \Lambda_n = \Lambda \cap R(x)_n$$

K-module Λ_n is finite - free $\forall n \in \mathbb{N}$

n	K-basis for Λ_n		
0	1		
1	$x_1 + x_2 + x_3 + \cdots = \sum_i x_i$		
2	$\sum_i x_i^2$	$\sum_{i<j} x_i x_j$	
3	$\sum_i x_i^3$	$\sum_{i \neq j} x_i^2 x_j$	$\sum_{i<j<k} x_i x_j x_k$

A <u>partition</u> is a weak composition

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

Such that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

For a partition λ write

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$$

Define $\ell = \underline{\text{length}}$ of λ

Often drop trailing 0's and write

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$$

Define

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell \qquad \text{"size of } \lambda\text{"}$$

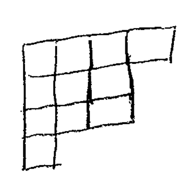
Say λ is a <u>partition of n</u> where $n = |\lambda|$

We often represent λ by its <u>Ferrar diagram</u> or Young diag

ex $\lambda = (4, 3, 3, 2)$

Ferrar diag

Young diag



Define

Par = set of all partitions

Par_n = set of all partitions of n $\quad n \in \mathbb{N}$

For $\lambda \in \mathrm{Par}$ define

$$m_\lambda = [x^\lambda] = \sum_{w \in S_{(\infty)} x^\lambda} w$$

"monomial sym functions"

Obs $\quad$ k-module Λ_n has k-basis

$m_\lambda \quad \lambda \in \mathrm{Par}_n$

Special cases of monomial sym functions:

λ	name for m_λ	desc of m_λ
(n)	power sym function p_n	$\sum_i x_i^n$
$(\underbrace{1,1,1,\ldots,1}_{n})$	elem sym function e_n	$\sum_{i_1<i_2<\cdots<i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$

Define

$$h_n = \sum_{\lambda \in Par_n} m_\lambda$$

"total sym functions"

$$= \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

Take $e_0 = p_0 = h_0 = 1$

Next general goal: turn Λ into a Hopf alg

that satisfies

- commutative, cocommutative, graded, connected
- p_n is primitive for $n \geq 1$.

The coproduct Δ of Λ

We define a K-alg morphism

$$\Delta : \Lambda \to \Lambda \otimes \Lambda$$

as follows:

Consider indets

$$x_1, x_2, \ldots$$

$$y_1, y_2, \ldots$$

The indets x_i have type x
y_i have type y

Consider sequence

$$z : \left(\begin{array}{cccc} x_1, & y_1, & x_2, & y_2, \ldots \\ \| & \| & \| & \\ z_1 & z_2 & z_3 \ldots & \end{array} \right)$$

View $\Lambda = \Lambda(x)$

We have K-alg iso

$$\Lambda \to \Lambda(z)$$

can: $m_\lambda \to m_\lambda$

We have k-alg morphism

$$\theta: \begin{array}{ccc} R(x) \otimes R(x) & \longrightarrow & R(z) \\ x_i \otimes 1 & \longrightarrow & x_i \\ 1 \otimes x_i & \longrightarrow & y_i \end{array}$$

θ is injective.

θ is not surj since its image does ~~not~~ contain

$$\sum_i x_i y_i$$

Consider usual $S_{(\infty)}$ action on $R(z)$.

Define a subgroup

$$S_{(\infty),(\infty)} = \{ \sigma \in S_{(\infty)} \mid \sigma \text{ leaves the type of each } x_i, y_i \text{ invar} \}$$

Consider

$$\{ f \in R(z) \mid f \text{ is fixed by all } \sigma \in S_{(\infty),(\infty)} \} \qquad (*)$$

$(*)$ is the image under θ of

$$\Lambda(x) \otimes \Lambda(x)$$

We define Δ to be the composition

$$\Delta : \Lambda \xrightarrow[\text{can}]{} \Lambda(z) \xrightarrow[\text{incl}]{} (*) \xrightarrow[\theta^{-1}]{} \Lambda(x) \otimes \Lambda(x) = \Lambda \otimes \Lambda$$

K-alg morph

So Δ is K-alg morphism.

Ex Find $\Delta(m_\lambda)$ for $\lambda = (1,1)$

$m_\lambda = e_2$

Δ sends

$$e_2 = \sum_{i<j} x_i x_j \xrightarrow[\text{can}]{} \sum_{i<j} z_i z_j = \sum_{i<j} x_i x_j + \sum_{i,j} x_i y_j + \sum_{i<j} y_i y_j$$

$$\xrightarrow{\theta^{-1}} \left(\sum_{i<j} x_i x_j \right) \otimes 1 + \left(\sum_i x_i \right) \otimes \left(\sum_j x_j \right) + 1 \otimes \left(\sum_{i<j} x_i x_j \right)$$

$$= e_2 \otimes 1 + e_1 \otimes e_1 + 1 \otimes e_2$$

<u>Notation</u> Given partitions λ, μ obtain a partition $\lambda \sqcup \mu$ by reordering the coords of the weak composition

$$(\lambda_1, \mu_1, \lambda_2, \mu_2, \dots)$$

For example

$$\begin{array}{llll} \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&& \end{array} \quad \sqcup \quad \begin{array}{lll}\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\\ \bullet&& \end{array} \quad = \quad \begin{array}{llll}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\\ \bullet&\bullet&\bullet&\\ \bullet&\bullet&&\\ \bullet&&& \end{array}$$

LEM For $\lambda \in Par$,

$$\Delta(m_\lambda) = \sum_{\substack{\mu, \nu \in Par \\ \mu \sqcup \nu = \lambda}} m_\mu \otimes m_\nu$$

pf Write

$$\lambda = (\lambda_1, \lambda_2, \dots)$$

We have

$$m_\lambda = \left[x_1^{\lambda_1} x_2^{\lambda_2} \cdots \right]$$

$$\downarrow \text{can}$$

$$\left[z_1^{\lambda_1} z_2^{\lambda_2} \cdots \right] \qquad (1)$$

$$\shortparallel \, ?$$

$$\sum_{\substack{\mu, \nu \in \mathrm{Par} \\ \mu \sqcup \nu = \lambda}} \theta\left(m_\mu \otimes m_\nu \right) \qquad (2)$$

Each of (1), (2) is a sum of mutually dist

monomials in $z_1, z_2, \ldots$

Given a monomial

$$z_1^{a_1} z_2^{a_2} \cdots \qquad (*)$$

show $(*)$ contributes to (1) iff $(*)$ contributes to (2)

$(*)$ contributes to (1) iff

$(a_1, a_2, \ldots)$ is perm of $(\lambda_1, \lambda_2, \ldots)$

For $\mu, \nu \in \mathrm{Par}$ st $\mu \sqcup \nu = \lambda$,

$(*)$ contributes to $\theta(m_\mu \otimes m_\nu)$ iff both

- the x-exponents among $(a_1, a_2 \ldots)$ give perm of $(\mu_1, \mu_2, \ldots)$
- the y-exponents - - - - - - $(\nu_1, \nu_2, \ldots)$

Therefore $(*)$ contributes to (1) iff $(*)$ contributes to (2)

Result follows. $\square$