Lec 15 Monday Oct 10

Check $\star$ is shuffle product

show: $\forall$ $f, g \in T(V)$ and $a \in T(V)$

$$\langle f \star g, a \rangle \overset{?}{=} \sum_{(a)} \langle f, a_1 \rangle \langle g, a_2 \rangle$$

WLOG f, g, a are words

$f = x_1 x_2 \cdots x_r$ $\quad g = y_1 y_2 \cdots y_s$ $\quad a = z_1 z_2 \cdots z_n$

$x_i, y_j, z_k \in X$

$f \star g =$ sum of shuffles of f, g

$$\langle f \star g, a \rangle = \begin{cases} 1 & \text{if } a \text{ is a shuffle of } f, g \\ 0 & \text{else} \end{cases}$$

Consider
$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2$$

$$\Delta(a) = \Delta(z_1 \cdots z_n) = \sum u_1 u_2 \cdots u_p \otimes v_1 v_2 \cdots v_q$$

sum over all words $u_1 \cdots u_p$ and $v_1 \cdots v_q$ such that $z_1 \cdots z_n$ is a shuffle of $u_1 \cdots u_p$ and $v_1 \cdots v_q$

For each summand $u_1 u_2 \cdots u_p \otimes v_1 v_2 \cdots v_q$

$$\langle f, u_1 u_2 \cdots u_p \rangle \langle g, v_1 v_2 \cdots v_q \rangle = \begin{cases} 1 & \text{if } f = u_1 \cdots u_p \text{ and } g = v_1 \cdots v_q \\ 0 & \text{else} \end{cases}$$

So

$$\sum_{(a)} \langle f, a_1 \rangle \langle g, a_2 \rangle \quad \text{becomes}$$

$$\begin{cases} 1 & \text{if } a \text{ is a shuffle of } f, g \\ 0 & \text{else} \end{cases}$$

So

$$\langle f ☆ g, a \rangle = \sum_{(a)} \langle f, a_1 \rangle \langle g, a_2 \rangle$$

Check Δ

Show: $\forall\ a, b \in T(V)$ and $f \in T(V)$

$$\underset{\substack{\uparrow \\ \text{concat}}}{\langle f, ab \rangle} \overset{?}{=} \underset{\substack{(f) \\ \uparrow \\ \text{wrt } \Delta}}{\sum} \langle f_1, a \rangle \langle f_2, b \rangle$$

WLOG f, a, b are words:

$f = x_1 x_2 \cdots x_n$ $\qquad a = y_1 \cdots y_r$ $\qquad b = z_1 \cdots z_s$

So

$ab = y_1 \cdots y_r z_1 \cdots z_s$

Obs

$$\langle f, ab \rangle = \begin{cases} 1 & \text{if } x_1 \cdots x_n = ab \\ 0 & \text{else} \end{cases}$$

obs

$$\sum_{(f)} \langle f_1, a\rangle \langle f_2, b\rangle =$$

$$\sum_{i=0}^{n} \underbrace{\langle x_1 \cdots x_i, a\rangle \langle x_{i+1} \cdots x_n, b\rangle}_{= \begin{cases} 1 & \text{if } x_1 \cdots x_i = a \text{ and } x_{i+1} \cdots x_n = b \\ 0 & \text{else} \end{cases}}$$

$$= \begin{cases} 1 & \text{if } \exists i \text{ st } x_1 \cdots x_i = a \text{ and } x_{i+1} \cdots x_n = b \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x_1 \cdots x_n = ab \\ 0 & \text{else} \end{cases}$$

So $\langle f, ab\rangle = \sum_{(f)} \langle f_1, a\rangle \langle f_2, b\rangle$

check S

$\forall\, f \in T(V)$ and $a \in T(V)$

$$\langle Sf, a \rangle \overset{?}{=} \langle f, Sa \rangle$$

WLOG f, a are words

$f = x_1 \cdots x_r \qquad a = y_1 \cdots y_s$

$$\langle Sf, a \rangle = \langle (-1)^r x_r \cdots x_1, \; y_1 \cdots y_s \rangle$$

$$= \begin{cases} (-1)^r & \text{if } x_r \cdots x_1 = y_1 \cdots y_s \\ 0 & \text{else} \end{cases}$$

$$\langle f, Sa \rangle = \langle x_1 \cdots x_r, \; (-1)^s y_s \cdots y_1 \rangle$$

$$= \begin{cases} (-1)^s & \text{if } x_1 \cdots x_r = y_s \cdots y_1 \\ 0 & \text{else} \end{cases}$$

— o —

Checking unit and counit is routine

$\square$

Special Case $\qquad$ V is K module basis x $\qquad$ "rank 1"

View $T(V) = K[x]$ $\qquad$ polys in X

	orig Hopf str.	dual Hopf str
mult	$x^i x^j = x^{i+j}$	$x^i \star x^j = \binom{i+j}{i} x^{i+j}$
unit	1	Same
comult	$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}$	$\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i}$
counit	$\varepsilon(x^n) = \delta_{n,0}$	same
antipode	$S(x^n) = (-1)^n x^n$	same

$\exists$ Hopf alg morphism

orig Hopf dual Hopf

$$k[x] \longrightarrow k[x]$$

$$x^n \longrightarrow n!\,x^n \qquad n \in \mathbb{N}$$

This morphism is a bijection provided $\mathbb{Q}$ is a subring of k

The shuffle algebra

We used the shuffle product to describe the restricted dual $T(V)^\circ$, where V is a finite-free K-module.

Now for any K-module V, we define the shuffle product

$$ш : T(V) \times T(V) \to T(V)$$

For $r, s \in \mathbb{N}$ define

$$ш : V^{\otimes r} \times V^{\otimes s} \to V^{\otimes(r+s)}$$

Motivation: take $r = s = 2$

Require of ш that $\forall\ a, b, c, d \in V$

$$\begin{aligned} a\otimes b \ ш\ c\otimes d = \ & a\otimes b\otimes c\otimes d \\ + \ & a\otimes c\otimes b\otimes d \\ + \ & a\otimes c\otimes d\otimes b \\ + \ & c\otimes a\otimes b\otimes d \\ + \ & c\otimes a\otimes d\otimes b \\ + \ & c\otimes d\otimes a\otimes b \end{aligned}$$

Not clear if Ш exists since dont have natural "basis" for $V^{\otimes 2}$

So we construct Ш in steps

Recall the usual mult in $T(V)$

$$m: \begin{array}{ccccc} V^{\otimes r} & \times & V^{\otimes s} & \longrightarrow & V^{\otimes(r+s)} \\ x & & y & \longrightarrow & x \otimes y \end{array}$$

Also for $n \in \mathbb{N}$, the sym group S_n acts on $V^{\otimes n}$ by permuting the tensor factors

ex $n=4$ $\sigma \in S_4$ sends $1 \to 2 \to 3 \to 4 \to 1$

action of σ on $V^{\otimes 4}$ sends

$$\underset{\substack{\| \\ v_1}}{a} \otimes \underset{\substack{\| \\ v_2}}{b} \otimes \underset{\substack{\| \\ v_3}}{c} \otimes \underset{\substack{\| \\ v_4}}{d} \longrightarrow \underset{\substack{\| \\ v_4 \\ \| \\ v_{\sigma^{-1}(1)}}}{d} \otimes \underset{\substack{\| \\ v_1 \\ \| \\ v_{\sigma^{-1}(2)}}}{a} \otimes \underset{\substack{\| \\ v_2 \\ \| \\ v_{\sigma^{-1}(3)}}}{b} \otimes \underset{\substack{\| \\ v_3 \\ \| \\ v_{\sigma^{-1}(4)}}}{c}$$

So $\sigma \in S_n$ acts on $V^{\otimes n}$ as

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \to v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

For $\sigma \in S_n$ and $0 \le r \le n$

call σ an <u>$(r, n-r)$-shuffle</u> whenever both

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r)$$

$$\sigma(r+1) < \sigma(r+2) < \cdots < \sigma(n)$$

We now define the shuffle product

$$ш : \begin{array}{ccccc} V^{\otimes r} & \times & V^{\otimes s} & \longrightarrow & V^{\otimes(r+s)} \\ x & & y & \longrightarrow & \displaystyle\sum_{\substack{\sigma \in S_{r+s} \\ \sigma \text{ is } (r,s)\text{-shuffle}}} \sigma(x \otimes y) \end{array}$$

One checks that ш turns $T(V)$ into an algebra with identity $1 \in k \equiv V^{\otimes 0}$. This algebra is commutative. Call it <u>shuffle algebra</u>

Next we extend this algebra str to Hopf alg str.

We now define coproduct Δ on $T(V)$

For $n \in \mathbb{N}$ and

$x_1 x_2 \cdots x_n \in V^{\otimes n}$

view

$x_1 x_2 \cdots x_n = x_1 \cdots x_i \otimes x_{i+1} \cdots x_n \in V^{i} \otimes V^{n-i}$ $\qquad 0 \leq i \leq n$

Define

$$\Delta : \begin{array}{ccc} T(V) & \to & T(V) \otimes T(V) \\ x_1 x_2 \cdots x_n & \to & \sum_{i=0}^{n} x_1 \cdots x_i \otimes x_{i+1} \cdots x_n \end{array}$$

Also let ε, S be same as for orig Hopf alg $T(V)$.

Then Δ, ε, S turn the shuffle algebra into a Hopf algebra.

One checks the required diagrams commute (ex)