Lec 14 Friday Oct 7

Given an algebra A

Turn A^* into a coalgebra using m^*, u^*

<u>Find Δ</u>

Try

$$\Delta : A^* \xrightarrow[m^*]{} (A \otimes A)^* \xrightarrow[can^{-1}]{} A^* \otimes A^*$$

(assuming can is a bijection)

For $f \in A^*$ write

$$\Delta(f) = \sum_{(f)} f_1 \otimes f_2$$

We have

$$f \longrightarrow m^*(f) = F \xleftarrow[can]{} \sum_{(f)} f_1 \otimes f_2$$

$\forall$ $a, b \in A$

$$F(a \otimes b) = \sum_{(f)} f_1(a) f_2(b)$$

$$\begin{aligned} &\| \\ &m^*(f)(a \otimes b) \\ &\| \\ &f(m(a \otimes b)) \\ &\| \\ &f(ab) \end{aligned}$$

Δ satisfies: For $f \in A^*$ and $a, b \in A$,

$$f(ab) = \sum_{(f)} f_1(a) f_2(b)$$

<u>Find ε</u>

try

$$\varepsilon : \quad A^* \xrightarrow{u^*} k^* \to k$$
$$h \to h(1)$$

For $f \in A^*$

$$f \longrightarrow u^*(f) \longrightarrow u^*(f)(1)$$
$$= f(u(1))$$
$$= f(1_A)$$

Get

$$\varepsilon : \quad A^* \to k$$
$$f \to f(1_A)$$

One checks the above Δ, ε turn A^* into a coalgebra.

Next suppose the alg A is graded:

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

Recall the graded dual

$$A^\circ = \bigoplus_{n \in \mathbb{N}} A_n^*$$

Turn this into a graded coalg using m^*, u^*

[possibly the coalg A° exists and the coalg A^* does not]

Find Δ by refining above approach

Recall the grading of $A \otimes A$:

$$A \otimes A = \sum_{n \in \mathbb{N}} (A \otimes A)_n$$

$$(A \otimes A)_n = \sum_{i=0}^{n} A_i \otimes A_{n-i}$$

For $n \in \mathbb{N}$

$$m\left((A \otimes A)_n\right) \subseteq A_n$$

So

$$A_n^* \xrightarrow[m^*]{} (A\otimes A)_n^* = \sum_{i=0}^{n} (A_i \otimes A_{n-i})^*$$

let

$$\Delta : A_n^* \xrightarrow[m^*]{} \sum_{i=0}^{n} (A_i \otimes A_{n-i})^* \xrightarrow[\text{can}^{-1}]{} \sum_{i=0}^{n} A_i^* \otimes A_{n-i}^*$$

(if can is bijection)

this gives a K-module hom

$$\Delta : A^\circ \to A^\circ \otimes A^\circ$$

provided that

$$A_r^* \otimes A_s^* \xrightarrow[\text{can}]{} (A_r \otimes A_s)^*$$

is a bijection for $r, s \in \mathbb{N}$

Find ε Take

$$\varepsilon : \quad A^{\circ} \to k$$
$$f \to f(1_A)$$

For $n \geq 1$ show

$$\varepsilon(A_n^*) = 0$$

For $f \in A_n^*$ show

$$\varepsilon(f) = 0$$

By constr

$$f(A_i) = 0 \quad \text{if } i \neq n \qquad i \in \mathbb{N}$$

$$1_A \in A_0$$

So $f(1_A) = 0$

Now $\varepsilon(f) = f(1_A) = 0$

One checks above Δ, ε turn A° into coalg.

By constr this coalg has grading

$$A^{\circ} = \bigoplus_{n \in \mathbb{N}} A_n^*$$

Given a Hopf alg H with antipode S

So $\forall a \in H$

$$\sum_{(a)} S(a_1)a_2 = \varepsilon(a)1_H = \sum_{(a)} a_1 S(a_2)$$

Coalg H turns H^* into alg

Alg H turns H^* into coalg (sometimes – let's assume it)

So H^* is bialg

Show $S^*: H^* \to H^*$ is an antipode that makes

H^* a Hopf alg.

For $f \in H^*$ show

$$\sum_{(f)} S^*(f_1) * f_2 \stackrel{?}{=} f(1_H)\mathbb{1} \stackrel{?}{=} \sum_{(f)} f_1 * S^*(f_2)$$

$\uparrow$

$\forall a \quad a \in H$

$$\left(\sum_{(f)} S^*(f_1) \star f_2 \right)(a) \overset{?}{=} f(1_H) \underbrace{\mathbb{1}(a)}_{\substack{\| \\ \varepsilon(a) 1_K \\ \| \\ \varepsilon(a)}}$$

$$\|$$

$$\sum_{(f)} \left(S^*(f_1) \star f_2 \right)(a)$$

$$\|$$

$$\sum_{(f)} \sum_{(a)} \underbrace{S^*(f_1)(a_1)}_{\substack{\| \\ f_1(S(a_1))}} f_2(a_2)$$

$$\|$$

$$\underbrace{\sum_{(a)} \left(\underbrace{\sum_{(f)} f_1(S(a_1)) f_2(a_2)}_{\substack{\| \\ f(S(a_1)\, a_2)}} \right)}$$

$$\|$$

$$f \left(\sum_{(a)} S(a_1)\, a_2 \right) \qquad \text{OK}$$

$$\|$$

$$f(\varepsilon(a) 1_H) = \varepsilon(a) f(1_H)$$

Now assume Hopf alg H is graded:

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Coalg H turns restricted dual H° into alg

Assume alg H turns H° into coalg

So H° is bialg.

Find antipode for H°

Antipode S for H satisfies

$$S(H_n) \subseteq H_n \qquad n \in \mathbb{N}$$

So $$S^*(H_n^*) \subseteq H_n^* \qquad n \in \mathbb{N}$$

So $$S^*(H^\circ) \subseteq H^\circ$$

S^* is antipode for H°

S^* turns H° into Hopf alg.

The Hopf alg H° has grading

$$H^\circ = \bigoplus_{n \in \mathbb{N}} H_n^*$$

Example The tensor algebra $T(V)$

Assume the K-module V is finite-free with basis X

View
$$V = \left\{ \sum_{x \in X} \alpha_x x \;\middle|\; \alpha_x \in K \right\}$$
"formal sums"

Recall tensor alg
$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

K module $T(V)$ is free with basis

$$x_1 x_2 \cdots x_n \qquad n \in \mathbb{N} \quad x_1, x_2, \ldots, x_n \in X \qquad (*)$$

$\uparrow$
word of length n

View
$$1 = \text{word of length } 0$$

$T(V)$ is a graded Hopf algebra of finite type

The restricted dual $T(V)^\circ$ is also a graded Hopf algebra of finite type.

We have k-module isomorphisms

$$T(V)^{\circ} \simeq \bigoplus_{n \in \mathbb{N}} (V^{\otimes n})^{*}$$

$$\simeq \bigoplus_{n \in \mathbb{N}} (V^{*})^{\otimes n}$$

$$= T(V^{*})$$

$$[V^{*} \simeq V]$$

$$\simeq T(V)$$

Via above isomorphisms we identify the k-modules

$$T(V)^{\circ}, \; T(V)$$

Under this identification the k-bilinear form

$$(\, , \,) \quad \begin{array}{ccccc} T(V)^{\circ} & \times & T(V) & \to & k \\ f & & a & \to & f(a) \end{array}$$

become a k-bilinear form on $T(V)$ wrt which

the basis $(*)$ is ortho-normal

The Kmodule $T(V)$ has the orig Hopf alg structure and a "dual" Hopf alg structure.

We now compare these structures

	orig Hopf str	dual Hopf str
mult	Concatenation of words	Convolution $\star$ is shuffle product
unit	1 = word of length 0	Same
comult	$\Delta(x) = x\otimes 1 + 1\otimes x \quad x\in X$ Δ is alg morph	$\Delta(x_1\cdots x_n) = \sum_{i=0}^{n} x_1\cdots x_i \otimes x_{i+1}\cdots x_n$
counit	$\varepsilon(x)=0 \quad x\in X$ ε is alg morph	Same
antipode	$S(x_1\cdots x_n) = (-1)^n x_n\cdots x_2 x_1$	Same

shuffle product:

Given words in X:

$x_1 x_2 \cdots x_r$ $\qquad$ $y_1 y_2 \cdots y_s$

A <u>shuffle</u> of these words is a word

$z_1 z_2 \cdots z_{r+s}$

st

- $x_1 x_2 \cdots x_r$ is a subword of $z_1 z_2 \cdots z_{r+s}$

  ie $\quad x_1 = z_{i_1}$, $\; x_2 = z_{i_2}$, $\;\vdots\;$, $x_r = z_{i_r}$ $\qquad i_1 < i_2 < \cdots < i_r$

- $y_1 y_2 \cdots y_s$ is a subword of $z_1 z_2 \cdots z_{r+s}$

  ie $\quad y_1 = z_{j_1}$, $\; y_2 = z_{j_2}$, $\;\vdots\;$, $y_s = z_{j_s}$ $\qquad j_1 < j_2 < \cdots < j_s$

- Above subwords dont overlap:

  $$\{ i_1, \ldots, i_r \} \cap \{ j_1 \ldots j_s \} = \emptyset$$

then

$x_1 x_2 \cdots x_r \star y_1 y_2 \cdots y_s =$ sum of shuffles for $x_1 x_2 \cdots x_r$ and $y_1 y_2 \cdots y_s$

For instance, given $a, b, c, d \in X$

$ab \star cd =$

$\underline{a}\ \underline{b}\ \underline{c}\ \underline{d}$

$+ \underline{a}\ \underline{c}\ \underline{b}\ \underline{d}$

$+ \underline{a}\ \underline{c}\ \underline{d}\ \underline{b}$

$+ \underline{c}\ \underline{a}\ \underline{b}\ \underline{d}$

$+ \underline{c}\ \underline{a}\ \underline{d}\ \underline{b}$

$+ \underline{c}\ \underline{d}\ \underline{a}\ \underline{b}$

So for $a = b = c = d$

$aa \star aa = 6\ aaaa$