Lec 13 Wed oct 5

Given K module V

Consider tensor alg $H = T(V)$

For $a \in H$ find

$$(S \star E)(a)$$

Recall
$E(a) = a \deg(a)$
for a homog

Case $a = v \in V$

v is prim

$$\Delta(v) = v \otimes 1 + 1 \otimes v$$

$$(S \star E)(v) = S(v)\underbrace{E(1)}_{=0} + \underbrace{S(1)}_{=1}\underbrace{E(v)}_{=v}$$

$$= v$$

Case $a = v_1 v_2$ $\quad v_1, v_2 \in V$

$$\Delta(a) = \Delta(v_1)\Delta(v_2)$$

$$= v_1 v_2 \otimes 1 + v_1 \otimes v_2 + v_2 \otimes v_1 + 1 \otimes v_1 v_2$$

$$(S \star E)(a) = S(v_1 v_2)\underbrace{E(1)}_{=0} + \underbrace{S(v_1)}_{=-v_1}\underbrace{E(v_2)}_{=v_2} + \underbrace{S(v_2)}_{=-v_2}\underbrace{E(v_1)}_{=v_1} + \underbrace{S(1)}_{=1}\underbrace{E(v_1 v_2)}_{=2 v_1 v_2}$$

$$= v_1 v_2 - v_2 v_1$$

LEM With above notation, for $n \geq 1$ and $v_1, v_2, \ldots, v_n \in V$,

$$(S \star E)(v_1 v_2 \cdots v_n) =$$

$$[\cdots[[v_1, v_2], v_3], \cdots v_n]$$

pf Use ind on n

$n = 1$ ✓

$n \geq 2$: show

$$(S \star E)(v_1 \cdots v_n) = [(S \star E)(v_1 \cdots v_{n-1}), v_n]$$

Write $a = v_1 v_2 \cdots v_{n-1}$

$p = v_n$ $\qquad$ p is prim

Obs

$$(S \star E)(v_1 \cdots v_n) = (S \star E)(ap)$$

$$= [(S \star E)(a), p] + \underset{\underset{\varepsilon(v_1)\varepsilon(v_2)\cdots\varepsilon(v_{n-1})}{\parallel}}{\varepsilon(a)} E(p)$$

$$\varepsilon(v_1) = 0, \quad \varepsilon(v_2) = 0, \quad \ldots, \quad \varepsilon(v_{n-1}) = 0$$

$$= [(S \star E)(v_1 \cdots v_{n-1}), v_n]$$

□

Duality

Given a k-module V

Recall the dual k-module

$$V^* = \mathrm{Hom}(V, k)$$

Ex Call V free whenever V is a direct product of copies of k

Call V finite free whenever V is a direct sum of finitely many copies of k.

Assume V is finite free

$$V = \underbrace{k \oplus k \oplus \cdots \oplus k}_{n}$$

Define

$$v_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i}}{1}, 0, \dots, 0) \qquad 1 \le i \le n \qquad \text{"basis for } V\text{"}$$

So

$$V = \sum_{i=1}^{n} k v_i \qquad \text{ds}$$

For $1 \le j \le n$ $\exists$ $v_j^* \in V^*$ st

$$v_j^*(v_i) = \delta_{ij} \qquad 1 \le i \le n$$

"dual basis for V^*"

One checks $\exists$ K-module iso $V \to V^*$ that sends $v_i \to v_i^*$ for $1 \le i \le n$

So V^* is finite free.

<u>Caution</u> For general K-modules V the dual V^* is not iso V

REV Given K-module V

$\exists$ K-bilinear form

$$(\, , \,) \qquad \begin{aligned} V^* \times V &\to K \\ f \quad v &\to f(v) \end{aligned}$$

Given K-modules U, V and a K-module hom

$$\varphi : U \to V$$

$\exists$ K-module hom

$$\varphi^* : V^* \to U^*$$

"the adjoint of φ"

st $\forall f \in V^*$ and $u \in U$

$$\varphi^*(f)(u) = f(\varphi(u))$$

ie

$$(\varphi^*(f), u) = (f, \varphi(u))$$

Given K-modules U, V

Consider dir sum $U \oplus V$

Compare $(U \oplus V)^*$, $U^* \oplus V^*$

For $f \in (U \oplus V)^*$

$f|_U \in U^*$, $f|_V \in V^*$

The map $(U \oplus V)^* \longrightarrow U^* \oplus V^*$

Can: $f \to (f|_U, f|_V)$

is iso of K-modules

Now consider tensor prod $U \otimes V$

Compare $(U \otimes V)^*$, $U^* \otimes V^*$

$\exists$ K-module hom

$$U^* \otimes V^* \longrightarrow (U \otimes V)^* \qquad (*)$$

Can: $f \otimes g \longrightarrow H$

st $H(u \otimes v) = f(u)\, g(v)$ $u \in U$, $v \in V$

The map $(*)$ is a bijection if U, V are finite free, but not in gen.

Assume the K-module V is graded:

$$V = \bigoplus_{n \in \mathbb{N}} V_n$$

For $f \in V^*$ $\quad f|_{V_n} \in V_n^*$

Define
$$V^\circ = \{ f \in V^* / \; f|_{V_n} \text{ nonzero for fin many } n \}$$

V° is K-submodule of V^*

The map
$$V^\circ \longrightarrow \bigoplus_{n \in \mathbb{N}} V_n^*$$
can
$$f \longrightarrow \bigoplus_{n \in \mathbb{N}} f|_{V_n}$$

is iso of K-modules

Call V° the <u>graded dual</u> of V

$\forall n \in \mathbb{N}$ let

$$V_n^\circ = \text{preimage of } V_n^* \text{ under can}$$
$$= \{ f \in V^* / \; f|_{V_i} = 0 \text{ for } i \neq n \}$$

By constr

$$V^\circ = \sum_{n \in \mathbb{N}} V_n^\circ \quad (\text{ds of K-modules})$$

Def: we say V has <u>finite type</u> whenever each V_n is finite-free

Given a coalg C

Recall $C^* = Hom(C, k)$ is an algebra with convolution prod $\star$

show $\star$ is the composition

$$\star : C^* \otimes C^* \xrightarrow{\text{can}} (C \otimes C)^* \xrightarrow{\Delta^*} C^*$$

check: $\forall f, g \in C^*$ show

$$f \otimes g \longrightarrow H \longrightarrow \Delta^*(H) \overset{?}{=} f \star g$$

$H(a \otimes b) = f(a) g(b)$

$\forall c \in C$

$$\Delta^*(H)(c) \overset{?}{=} (f \star g)(c)$$

$$\Delta^*(H)(c) = H(\Delta(c)) = \sum_{(c)} \underbrace{H(c_1 \otimes c_2)}_{= f(c_1) g(c_2)}$$

$$(f \star g)(c) = \sum_{(c)} f(c_1) g(c_2)$$

ok.

For the algebra C^* show the unit u is the comp

$$u: \quad K \xrightarrow[\text{can}]{} K^* \xrightarrow[\varepsilon^*]{} C^*$$

check

$$1 \xrightarrow[f(1)=1]{} f \longrightarrow \varepsilon^* f \overset{?}{=} \mathbb{1}$$

$\forall\, c \in C$ show

$$\varepsilon^* f\,(c) \overset{?}{=} \mathbb{1}(c)$$

$$\varepsilon^* f(c) = f(\varepsilon(c)) = \underbrace{\varepsilon(c)\, \underset{=1}{f(1)}} = \varepsilon(c)$$

$$\mathbb{1}(c) = \varepsilon(c)\, 1_K = \varepsilon(c)$$

ok

Next assume only C is graded:

$$C = \bigoplus_{n \in \mathbb{N}} C_n$$

We saw C^* is an algebra

What about the graded dual C°?

We show C° is a subalgebra of C^*

Show $\mathbb{1} \in C^\circ$

Recall
$$\mathbb{1} : \begin{array}{l} C \to K \\ c \to \varepsilon(c) \end{array}$$

For $n \in \mathbb{N}$ find $\mathbb{1}/C_n$

For $n \geq 1$ $\varepsilon(C_n) = 0$

So $\mathbb{1}/C_n = 0$

So $\mathbb{1} \in C_0^\circ \subseteq C^\circ$ ✓

Show C° is closed under $\star$

For $r, s \in \mathbb{N}$ show
$$C_r^\circ \star C_s^\circ \subseteq C_{r+s}^\circ$$

Given $f \in C_r^\circ$ $g \in C_s^\circ$

show $f \star g \in C_{r+s}^\circ$

For $n \in \mathbb{N}$ st $n \neq r+s$ show

$$(f \star g) / C_n = 0$$

$\forall\, c \in C_n$

$$(f \star g)(c) = \sum_{(c)} f(c_1)\, g(c_2)$$

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 \in C_0 \otimes C_n + C_1 \otimes C_{n-1} + \cdots + C_n \otimes C_0$$

WLOG, for each term $c_1 \otimes c_2$

c_1, c_2 are homog with $\deg(c_1) + \deg(c_2) = n$

Since $n \neq r+s$,

$\deg(c_1) \neq r$ or $\deg(c_2) \neq s$

So $f(c_1) = 0$ or $g(c_2) = 0$

So $f(c_1)\, g(c_2) = 0$

So $$(f \star g)(c) = \sum_{(c)} \underbrace{f(c_1)\, g(c_2)}_{= 0}$$

$$= 0$$

So $(f \star g) \,/\, C_n = 0$

So $f \star g \in C^0_{r+s}$

We have shown

- C^0 is a subalg of $C^\vee$
- Algebra C^0 has a grading

$$C^0 = \bigoplus_{n \in \mathbb{N}} C^0_n$$

Next we consider the ∞ - situation