Lec 12 Monday Oct 3 10/3/16

1

Thm Given a commutative Hopf alg H

Consider $id \in End(H)$

then for $r, s \in \mathbb{Z}$

$$id^{\star r} \circ id^{\star s} = id^{\star(rs)}$$

pf For $r, s \geq 0$ done by LEM IV

Show for $r, s \geq 0$

$$\underbrace{id^{\star(-r)}}_{\substack{\parallel \\ S^{\star r}}} \circ id^{\star s} \overset{?}{=} \underbrace{id^{\star(-rs)}}_{\substack{\parallel \\ S^{\star(rs)}}}$$

S is alg morph since H is com

$S^{\star r}$ is alg morph by LEM I

By LEM III (with $\alpha = S^{\star r}$, $\gamma = id$)

$$\begin{aligned} S^{\star r} \circ id^{\star s} &= \alpha \circ id^{\star s} \circ \gamma \\ &= (\underbrace{\alpha \circ id \circ \gamma}_{\parallel \, \alpha})^{\star s} \\ &= (S^{\star r})^{\star s} \\ &= S^{\star(rs)} \quad \checkmark \end{aligned}$$

Fn $r, s \geq 0$ show

$$id^{\star r} \circ \underbrace{id^{\star(-s)}}_{\substack{\parallel \\ S^{\star s}}} \overset{?}{=} \underbrace{id^{\star(-rs)}}_{\substack{\parallel \\ S^{\star(rs)}}}$$

Recall

$id^{\star r}$ is alg morph

Apply LEM III (with $\alpha = id^{\star r}$, $\gamma = id$)

$$\begin{aligned}
id^{\star r} \circ S^{\star s} &= \alpha \circ S^{\star s} \circ \gamma \\
&= \big(\underbrace{\alpha \circ S \circ \gamma}_{\substack{\parallel \\ \alpha \circ S \\ \parallel \\ \text{inv of } \alpha \text{ wrt } \star \text{ by LEM II} \\ \parallel \\ S^{\star r}}}\big)^{\star s} \\
&= \left(S^{\star r}\right)^{\star s} \\
&= S^{\star(rs)} \qquad \text{OK}
\end{aligned}$$

For $r, s \geq 0$ show

$$\underbrace{id^{\star(-r)}}_{\substack{\shortparallel \\ S^{\star r}}} \circ \underbrace{id^{\star(-s)}}_{\substack{\shortparallel \\ S^{\star s}}} = id^{\star(rs)}$$

$\uparrow$
algmorph

By LEM III (with $\alpha = S^{\star r}$, $\gamma = id$)

$$S^{\star r} \circ S^{\star s} = \alpha \circ S^{\star s} \circ \gamma$$

$$= (\underbrace{\alpha \circ S \circ \gamma}_{\substack{\shortparallel \\ \alpha \circ S \\ \shortparallel \\ \text{inv of } \alpha \text{ w-t } \star \\ \shortparallel \\ id^{\star r}}})^{\star s}$$

$$= (id^{\star r})^{\star s}$$

$$= id^{\star(rs)}$$

✓

$\square$

We now do a "co" version of the previous result.

LEM I Given

k-bialg H

cocommutative k-coalg C

coalg morphisms

$f : C \to H$, $g : C \to H$

then

$f \star g : C \to H$

is coalg morphism.

pf Given

$$\begin{array}{ccc} C & \xrightarrow{f} & H \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_H} \\ C \otimes C & \xrightarrow[f \otimes f]{} & H \otimes H \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & H \\ {\scriptstyle \varepsilon_C}\downarrow & & \downarrow{\scriptstyle \varepsilon_H} \\ K & \xrightarrow[id]{} & K \end{array}$$

and similar for g

show

$$\begin{array}{ccc} C & \xrightarrow{f \star g} & H \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_H} \\ C \otimes C & \xrightarrow[(f \star g) \otimes (f \star g)]{} & H \otimes H \end{array}$$

Chase $c \in C$ around diagram.

$$\Delta_H\Big((f * g)(c)\Big) \overset{?}{=} \sum_{(c)} (f * g)(c_1) \otimes (f * g)(c_2)$$

$$LHS = \Delta_H\Big(\sum_{(c)} f(c_1)\, g(c_2)\Big)$$

$$= \sum_{(c)} \underbrace{\Delta_H(f(c_1))}_{\sum_{(c_1)} f((c_1)_1) \otimes f((c_1)_2)} \; \underbrace{\Delta_H(g(c_2))}_{\sum_{(c_2)} g((c_2)_1) \otimes g((c_2)_2)}$$

$$\Big[\sum_{(c)} \Delta(c_1) \otimes \Delta(c_2) = \sum_{(c)} c_1 \otimes c_2 \otimes c_3 \otimes c_4 \Big]$$

$$= \sum_{(c)} \big(f(c_1) \otimes f(c_2)\big)\big(g(c_3) \otimes g(c_4)\big)$$

$$= \sum_{(c)} f(c_1)\, g(c_3) \otimes f(c_2)\, g(c_4)$$

$$\Big[\text{Since } C \text{ is cocom} \quad \sum_{(c)} c_1 \otimes c_2 \otimes c_3 \otimes c_4 = \sum_{(c)} c_1 \otimes c_3 \otimes c_2 \otimes c_4 \Big]$$

$$= \sum_{(c)} f(c_1)\, g(c_2) \otimes f(c_3)\, g(c_4)$$

$$RHS = \sum_{(c)} \left(\sum_{(c_1)} f((c_1)_1)\, g((c_1)_2) \right) \otimes \left(\sum_{(c_2)} f((c_2)_1)\, g((c_2)_2) \right)$$

$$= \sum_{(c)} f(c_1)\, g(c_2) \otimes f(c_3)\, g(c_4)$$

ok

Show

$$\begin{array}{ccc} C & \xrightarrow{f \star g} & H \\ \varepsilon_C \downarrow & & \downarrow \varepsilon_H \\ K & \xrightarrow[id]{} & K \end{array}$$

$\forall c \in C$

$$\varepsilon_H((f \star g)(c)) \overset{?}{=} \varepsilon_C(c)$$

$$\| $$

$$\varepsilon_H\left(\sum_{(c)} f(c_1)\, g(c_2) \right)$$

$$\| $$

$$\underbrace{\sum_{(c)} \underbrace{\varepsilon_H(f(c_1))}_{= \varepsilon_C(c_1)}\, \underbrace{\varepsilon_H(g(c_2))}_{= \varepsilon_C(c_2)}}_{=}$$

$$\varepsilon_C\left(\underbrace{\sum_{(c)} c_1\, \varepsilon_C(c_2)}_{= c} \right)$$

$$\| $$

$$\varepsilon_C(c) \qquad \text{ok}$$

$\square$

LEM II Given

Hopf alg H with antipode S

cocommutative k-coalg C

Coalg morph

$$f \quad C \longrightarrow H \qquad (*)$$

then

$$C \xrightarrow{f} H \xrightarrow{S} H \qquad (**)$$

is coalg morph. Moreover $*$, $**$ are inverses wrt $\star$

pf Check $**$ is coalg morph.

show

$$\begin{array}{ccc} C & \xrightarrow{S \circ f} & H \\ \Delta_C \downarrow & & \downarrow \Delta_H \\ C \otimes C & \xrightarrow[(S \circ f) \otimes (S \circ f)]{} & H \otimes H \end{array}$$

$\forall c \in C$

$$\Delta_H(S(f(c))) \stackrel{?}{=} \sum_{(c)} S(f(c_1)) \otimes S(f(c_2))$$

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2$$

$$= \sum_{(c)} c_2 \otimes c_1 \qquad \text{since } C \text{ is cocom}$$

$$\Delta_H(f(c)) = \sum_{(c)} f(c_2) \otimes f(c_1)$$

$$\Delta_H(S(f(c))) = \sum_{(c)} S(f(c_1)) \otimes S(f(c_2))$$

ok

Show

$$\begin{array}{ccc} C & \xrightarrow{S \circ f} & H \\ \varepsilon_C \downarrow & & \downarrow \varepsilon_H \\ k & \xrightarrow{id} & k \end{array}$$

$\forall c \in C$ $\quad \varepsilon_H(S(f(c))) \stackrel{?}{=} \varepsilon_C(c)$

$\|$ S is antipode for H

$\varepsilon_H(f(c))$

$\|$ f is coalg morph

$\varepsilon_C(c)$

check $*_1$ $**$ are inverses wrt $\star$

$$f \star (S \circ f) \overset{?}{=} \mathbb{1} \overset{?}{=} (S \circ f) \star f$$

$\uparrow$

$\forall c \in C$

$$\sum_{(c)} f(c_1)\, S(f(c_2)) \overset{?}{=} \varepsilon(c)\, 1_H$$

Write $g = f(c)$

So
$$\Delta(g) = \sum_{(g)} g_1 \otimes g_2 = \sum_{(c)} f(c_1) \otimes f(c_2)$$

and
$$\varepsilon(g) = \varepsilon(f(c)) = \varepsilon(c)$$

Also since S is antipode for H,

$$\sum_{(g)} g_1\, S(g_2) = \varepsilon(g)\, 1_H$$

$$\| \qquad\qquad\qquad \|$$

$$\sum_{(c)} f(c_1)\, S(f(c_2)) \qquad \varepsilon(c)\, 1_H$$

ok

$\square$

LEM III and its "co" version are the same.

LEM IV Given

cocommutative bialg H

Consider $id \in End(H)$

Then for $r, s \geq 0$

$$id^{\star r} \circ id^{\star s} = id^{\star (rs)}$$

pf id is coalg morph

$id \star id$ is coalg morph by LEM I

..

$id^{\star s}$ is coalg morph

By LEM III with

$$\alpha = id, \quad \gamma = id^{\star s}$$

$$\alpha \circ id^{\star r} \circ \gamma = id^{\star r} \circ id^{\star s}$$

$$\| $$

$$\underbrace{(\underbrace{\alpha \circ id \circ \gamma}_{= id^{\star s}})^{\star r}}_{= id^{\star (rs)}}$$

$\square$

Thm Given a cocommutative Hopf alg H

Consider $id \in End(H)$

Then for $r, s \in \mathbb{Z}$,

$$id^{\star r} \circ id^{\star s} = id^{\star (rs)}$$

pf For $r, s \geq 0$ done by LEM IV

show for $r, s \geq 0$

$$\underbrace{id^{\star(-r)}}_{= S^{\star r}} \circ id^{\star s} = \underbrace{id^{\star(-rs)}}_{= S^{\star(rs)}}$$

Recall $id^{\star s}$ is coalg morphism

By LEM III (with $\alpha = id$, $\gamma = id^{\star s}$)

$$S^{\star r} \circ id^{\star s} = \alpha \circ S^{\star r} \circ \gamma$$

$$= (\underbrace{\alpha \circ S \circ \gamma}_{= S \circ \gamma})^{\star r}$$

$S \circ \gamma$ = inverse of γ wrt $\star$ = $S^{\star s}$

$$= S^{\star(rs)} \quad \checkmark$$

For $r, s \geq 0$ show

$$id^{\star r} \circ id^{\star(-s)} \overset{?}{=} id^{\star(-rs)}$$

$$id^{\star(-s)} = S^{\star s}, \qquad id^{\star(-rs)} = S^{\star(rs)}$$

Since H is cocom,

S is coalg morph.

So $S^{\star s}$ is colg morph

Apply LEM (with $\alpha = id$, $\gamma = S^{\star s}$)

$$\begin{aligned} id^{\star r} \circ S^{\star s} &= \alpha \circ id^{\star r} \circ \gamma \\ &= (\alpha \circ id \circ \gamma)^{\star r} \\ &= (S^{\star s})^{\star r} \\ &= S^{\star(rs)} \quad \checkmark \end{aligned}$$

For $r, s \geq 0$ show

$$\begin{array}{ccccc} id^{\star(-r)} & \circ & id^{\star(-s)} & \overset{?}{=} & id^{\star(rs)} \\ \| & & \| & & \\ S^{\star r} & & S^{\star s} & & \\ & & \uparrow & & \\ & & \text{coalg morph} & & \end{array}$$

By LEM III (with $\alpha = id$, $\gamma = S^{\star s}$)

$$\begin{aligned} S^{\star r} \circ S^{\star s} &= \alpha \circ S^{\star r} \circ \gamma \\ &= (\underbrace{\alpha \circ S \circ \gamma}_{\substack{\| \\ S \circ \gamma \\ \| \\ \text{inv of } \gamma \text{ rel } \star \\ \| \\ id^{\star s}}})^{\star r} \\ &= id^{\star(rs)} \end{aligned}$$

$\square$

LEM Given cocommutative graded Hopf alg

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Define $E \in \mathrm{End}(H)$ by

$$E(a) = \underset{\substack{\shortparallel \\ \deg(a)}}{n}\, a \qquad a \in H_n$$

Then $\forall a \in H$, each of

$$\underset{\substack{\shortparallel \\ x}}{(S \star E)(a)}, \qquad (E \star S)(a)$$

is primitive.

pf Show x is primitive.

$$\Delta(x) \overset{?}{=} x \otimes 1 + 1 \otimes x$$

$$x = (S \star E)(a)$$

$$= \sum_{(a)} S(a_1) E(a_2) \qquad (\bullet)$$

WLOG $a \in H_n$

$$\Delta(a) \in \sum_{i=0}^{n} H_i \otimes H_{n-i}$$

Write

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2$$

WLOG each term $a_1 \otimes a_2 \in H_i \otimes H_{n-i}$ some i

$$E(a_2) = n-i = \deg(a_2)$$

So $(\bullet)$ becomes

$$X = \sum_{(a)} S(a_1)\, a_2 \deg(a_2)$$

$$\Delta(X) = \sum_{(a)} \Delta(S(a_1))\, \Delta(a_2) \deg(a_2)$$

$$\Delta(S(a_1)) = \sum_{(a_1)} S((a_1)_2) \otimes S((a_1)_1)$$

$$\Delta(a_2) = \sum_{(a_2)} (a_2)_1 \otimes (a_2)_2$$

WLOG $(a_2)_1$, $(a_2)_2$ homog

$$\deg((a_2)_1) + \deg((a_2)_2) = \deg(a_2)$$

$$\Delta(a_2)\deg(a_2) = \sum_{(a_2)} (a_2)_1 \otimes (a_2)_2 \left(\deg((a_2)_1) + \deg((a_2)_2)\right)$$

So far

$$\Delta(x) = \sum_{(a)} \sum_{(a_1)} \sum_{(a_2)} S((a_1)_2)(a_2)_1 \otimes S((a_1)_1)(a_2)_2 \left(\begin{array}{l} \deg((a_2)_1) + \\ \deg((a_2)_2) \end{array} \right) \quad (**)$$

obs

$$\sum_{(a)} \sum_{(a_1)} \sum_{(a_2)} (a_1)_1 \otimes (a_1)_2 \otimes (a_2)_1 \otimes (a_2)_2$$

$$= \sum_{(a)} \Delta(a_1) \otimes \Delta(a_2)$$

$$= \sum_{(a)} a_1 \otimes a_2 \otimes a_3 \otimes a_4$$

↑ we can permute the tensor factors by cocom

(**) becomes

$$\Delta(x) = \sum_{(a)} S(a_1) a_2 \otimes S(a_3) a_4 \left(\deg(a_2) + \deg(a_4) \right)$$

$$\sum_{(a)} S(a_1)\, a_2 \otimes S(a_3)\, a_4 \quad \deg(a_2)$$

$$\left[\sum_{(b)} S(b_1)\, b_2 = \varepsilon(b) 1_H \right]$$

$$= \sum_{(a)} S(a_1)\, a_2 \deg(a_2) \otimes \varepsilon(a_3)\, 1_H$$

$$\left[\sum_{(a)} a_1 \otimes a_2 \otimes \varepsilon(a_3) = \sum_{(a)} a_1 \otimes a_2 \otimes 1 \right]$$

$$= \sum_{(a)} \underbrace{S(a_1)\, a_2 \deg(a_2)}_{= x} \otimes 1$$

$$= x \otimes 1$$

Similarly

$$\sum_{(a)} S(a_1)\, a_2 \otimes S(a_3)\, a_4 \deg(a_4) = 1 \otimes x.$$

So

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

LEM For above H and primitive $a \in H$,

$$(S \ast E)(a) = E(a) = (E \ast S)(a)$$

pf $\Delta(a) = a \otimes 1 + 1 \otimes a$

so $$(S \ast E)(a) = \sum_{(a)} S(a_1)\, E(a_2)$$

$$= S(a)\underbrace{E(1)}_{\substack{= 1\,\deg(1) \\ = 0}} + \underbrace{S(1)}_{=1} E(a)$$

$$= E(a)$$

□

LEM For above H

For $a \in H$, prim $p \in H$

$$(S \star E)(ap) = \underbrace{\left[(S \star E)(a),\ p \right]}_{\text{commutator}} + \varepsilon(a)\, E(p)$$

pf WLOG a, p are homog

$$\Delta(ap) = \underbrace{\Delta(a)}_{= \sum_{(a)} a_1 \otimes a_2} \ \underbrace{\Delta(p)}_{= p \otimes 1 + 1 \otimes p}$$

$$= \sum_{(a)} \left(a_1 p \otimes a_2 + a_1 \otimes a_2 p \right)$$

$$(S \star E)(ap) = \sum_{(a)} \Big(\underbrace{S(a_1 p)}_{= S(p) S(a_1) = -p\, S(a_1)} E(a_2) + S(a_1)\, E(a_2 p) \Big)$$

$$\left[\begin{array}{l} \text{WLOG each } a_2 \text{ is homog} \\ E(a_2 p) = a_2 p \deg(a_2 p) \\ \qquad\qquad\qquad \overset{\shortparallel}{\deg(a_2) + \deg(p)} \\ \qquad = E(a_2)p + a_2 E(p) \end{array}\right]$$

$$= \sum_{(a)} \Big(S(a_1) E(a_2) p - p S(a_1) E(a_2) \Big)$$

$$+ \sum_{(a)} S(a_1) a_2 E(p)$$

$$= \Big[\underbrace{\sum_{(a)} S(a_1) E(a_2)}_{(S \star E)(a)}, p \Big] + \Big(\underbrace{\sum_{(a)} S(a_1) a_2}_{\overset{\shortparallel}{\varepsilon(a) 1_H}} \Big) E(p)$$

$$= \big[(S \star E)(a), p \big] + \varepsilon(a) E(p)$$

$\square$

Prop. Given a graded, connected, cocom Hopf alg

$$H = \oplus_{n \in \mathbb{N}} H_n$$

Assume that for $n \in \mathbb{N}$

$\underbrace{1+1+\cdots+1}_{n}$ is invertible in K

[so $\mathbb{Q}$ is subring of K]

Then the alg H is generated by its primitive elements P.

pf For $n \in \mathbb{N}$ show $H_n \subseteq \langle P \rangle$

By induction on n

$n=0$: $H_0 = K\, 1_H \subseteq \langle P \rangle$

$n=1$: $H_1 \subseteq P \subseteq \langle P \rangle$

$n \geq 2$: For $a \in H_n$ show $a \in \langle P \rangle$

Write $\Delta(a) = \sum_{(a)} a_1 \otimes a_2 \in \sum_{i=0}^{n} H_i \otimes H_{n-i}$

Recall
$$\Delta(a) - a \otimes 1 - 1 \otimes a \in \sum_{i=1}^{n-1} H_i \otimes H_{n-i}$$

So

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum_{(a)'} a_1 \otimes a_2$$

WLOG for each term $a_1 \otimes a_2$

each of a_1, a_2 are homog with

$1 \le \deg(a_1),\ \deg(a_2) \le n$

$\deg(a_1) + \deg(a_2) = n$

Define $p = (S \star E)(a) \quad \in P$

$$p = \sum_{(a)} S(a_1)\, E(a_2)$$

$$= \underbrace{S(a)\, E(1)}_{E(1) = 1\deg(1) = 0} + \underbrace{S(1)}_{=1}\, \underbrace{E(a)}_{= an} + \sum_{(a)'} S(a_1)\, \underbrace{E(a_2)}_{= a_2 \deg(a_2)}$$

$$= na + \sum_{(a)'} S(a_1)\, a_2 \deg(a_2)$$

So

$$p - na = \sum_{(a_j)'} s(a_j)\, a_2 \deg(a_2)$$

[Recall $s(H_i) \subseteq H_i \quad \forall i$

$s(a_j)$ is homog, same degree as a_j]

$$\subseteq \sum_{i=1}^{n-1} H_i H_{n-i}$$

[by ind $H_j \subseteq \langle P \rangle$ for $1 \leq j \leq n-1$]

$$\subseteq \langle P \rangle$$

Now

$$na \in \langle P \rangle$$

But $n^{-1} \in K$ so

$$a \in \langle P \rangle$$

$\square$