Lec 11 Friday Sept 30

DEF A <u>Lie algebra over k</u> is a k-module

L together with a k-bilinear map

$$[\,,\,] : \quad L \times L \to L$$
$$a, b \to [a,b]$$

st

$$[a,a] = 0 \qquad a, b, c \in L$$

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

"Jacobi"

Ex Given an (assoc) k-alg A

A becomes a Lie alg over k with

$$[a,b] = ab - ba \qquad a, b \in A$$

Ex Given a bialg H over k

Consider the set $P = P(H)$ of primitive elements in H.

then P is a Lie alg over k with

$$[a,b] = ab - ba$$

Next goal: Given a Lie algebra L over k

Show how the univ enveloping algebra $U(L)$ becomes a Hopf algebra.

Recall the tensor alg $T(L) = \bigoplus_{n \in \mathbb{N}} L^{\otimes n}$ is a Hopf alg over k with

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
$$\varepsilon(x) = 0 \qquad x \in L$$
$$S(x) = -x$$

let J = 2-sided ideal of $T(L)$ gen by

$$xy - yx - [x,y] \qquad x, y \in L$$

Consider quotient k-module

$$U(L) = T(L)/J$$

Turns out

$$(L^{\otimes 0} + L^{\otimes 1}) \cap J = 0 \qquad (\text{REV})$$

So the map

$$i : L \xrightarrow[\text{incl}]{} T(L) \xrightarrow[\text{can}]{} U(L)$$

is injective

We identify L with its image under i

So L becomes a k-submodule of $U(L)$.

So far, $U(L)$ is a k-algebra that is generated by L.

Next we turn $U(L)$ into a Hopf alg.

<u>Show J is a coideal of $T(L)$</u>

show $\Delta(J) \overset{?}{\subseteq} J \otimes T(L) + T(L) \otimes J$

$\forall\, x, y \in L$,

$$\Delta(xy - yx - [x,y]) = \underbrace{\Delta(x)\Delta(y) - \Delta(y)\Delta(x)}_{\parallel \atop (xy-yx)\otimes 1 + 1\otimes(xy-yx)} - \underbrace{\Delta([x,y])}_{\parallel \atop [x,y]\otimes 1 + 1\otimes[x,y]}$$

$$= (xy - yx - [x,y]) \otimes 1 + 1 \otimes (xy - yx - [x,y])$$

$$\in J \otimes T(L) + T(L) \otimes J \quad ✓$$

show $\varepsilon(J) \overset{?}{=} 0$

$\forall\, x, y \in L$

$$\varepsilon(xy - yx - [x,y]) = \underset{\parallel \atop 0}{\varepsilon(x)}\underset{\parallel \atop 0}{\varepsilon(y)} - \underset{\parallel \atop 0}{\varepsilon(y)}\underset{\parallel \atop 0}{\varepsilon(x)} - \underset{\parallel \atop 0}{\varepsilon([x,y])}$$

$$= 0 \quad ✓$$

J is coideal ✓

?

Show $S(J) \subseteq J$

$\forall\, x, y \in L$

$$S(xy - yx - [x,y]) = \underset{-y}{\underset{\parallel}{S(y)}}\,\underset{-x}{\underset{\parallel}{S(x)}} - \underset{-x}{\underset{\parallel}{S(x)}}\,\underset{-y}{\underset{\parallel}{S(y)}} - \underset{-[x,y]}{\underset{\parallel}{S([x,y])}}$$

$$= -\left(xy - yx - [x,y]\right)$$

$$\in J$$

ok

By these comments Δ, ε, S induce on $U(L)$

a Hopf alg str st

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
$$\varepsilon(x) = 0 \qquad \forall x \in L \subseteq U(L)$$
$$S(x) = -x$$

this Hopf alg is cocommutative.

Each element of L is primitive in $U(L)$

So $L \subseteq P(H)$ where $H = U(L)$

$\uparrow$

set of prim elements in H

bialgebras over K		Lie algebras over K	
H	$\longrightarrow$	$P(H)$	$*$
$U(L)$	$\longleftarrow$	L	$**$

$*$, $**$ not inverses in gen

Given commutative Hopf alg H with antipode S.

Recall for $f, g \in \mathrm{End}(H)$

$$(f \star g)(x) = \sum_{(x)} f(x_1)\, g(x_2) \qquad x \in H$$

Consider $id \in \mathrm{End}(H)$

For $n \in \mathbb{Z}$ consider $id^{\star n}$

this means

$$\underbrace{id \star id \star \cdots \star id}_{n} \qquad \text{if } n > 0$$

$$\mathbb{1} \qquad \text{if } n = 0$$

$$\left[\mathbb{1}(x) = \varepsilon(x) 1_H \qquad x \in H \right]$$

$$\underbrace{S \star S \star \cdots \star S}_{|n|} \qquad \text{if } n < 0$$

[recall S, id are inverses w.r.t $\star$]

Next goal: For $r, s \in \mathbb{Z}$ find composition

$$H \xrightarrow{id^{\star r}} H \xrightarrow{id^{\star s}} H$$

We will take steps I – IV

LEM I Given:

k-bialgebra H

commutative k-algebra A

algebra morphisms

$f: H \to A$, $\quad g: H \to A$

then $f \star g : H \to A$ is algebra morphism.

pf

$$(f \star g)(1_H) \overset{?}{=} 1_A$$

$$[\Delta(1_H) = 1_H \otimes 1_H]$$

$$\| $$

$$\underbrace{\underset{1_A}{\overset{\|}{f(1_H)}}\ \underset{1_A}{\overset{\|}{g(1_H)}}}_{\| \atop 1_A} \quad \text{ok}$$

$\forall\ x, y \in H$

$$(f \star g)(xy) \overset{?}{=} \underbrace{(f \star g)(x)}_{\displaystyle = \sum_{(x)} f(x_1)\, g(x_2)} \ \underbrace{(f \star g)(y)}_{\displaystyle = \sum_{(y)} f(y_1)\, g(y_2)}$$

[A is com]

$$= \sum_{(x)} \sum_{(y)} f(x_1)\, f(y_1)\ g(x_2)\, g(y_2)$$

[f, g are alg morphisms]

$$= \sum_{(x)} \sum_{(y)} f(x_1 y_1)\ g(x_2 y_2)$$

$$\left[\Delta(xy) = \Delta(x)\, \Delta(y) = \sum_{(x)} \sum_{(y)} x_1 y_1 \otimes x_2 y_2 \right]$$

$$= (f \star g)(xy)$$

ok

$\square$

LEM II Given:

Hopf alg H with antipode S

commutative k-alg A

algebra morphism

$$f : H \to A \qquad *$$

Then $\qquad **$

$$H \xrightarrow{S} H \xrightarrow{f} A$$

is an alg morphism. Moreover $*, **$ are inverses wrt $\star$.

pf Check $**$ is alg morphism

$$f(S(1_H)) = f(1_H) = 1_A \quad \checkmark$$

$\forall\, x, y \in H$

$$f(S(xy)) = f(S(y)\,S(x)) = f(S(y))\,f(S(x))$$

[A com]

$$= f(S(x))\,f(S(y))$$

ok

Show $*_1\ **$ are inverses wrt $\star$

check $f \star (f \circ S) \overset{?}{=} \mathbb{1}$

$\forall x \in H$

$$\big(f \star (f \circ S)\big)(x) \overset{?}{=} \mathbb{1}(x) = \varepsilon(x)\, 1_A$$

$$= \sum_{(x)} f(x_1)\, f(S(x_2))$$

$$= \underbrace{f\Big(\underbrace{\sum_{(x)} x_1 S(x_2)}_{= \varepsilon(x) 1_H}\Big)}_{= \varepsilon(x) \underbrace{f(1_H)}_{= 1_A}}$$

ok

Similarly

$$(f \circ S) \star f = \mathbb{1}$$

$\square$

LEM III Given

algebras A, A'

coalgebras C, C'

alg morph $\alpha : A \to A'$

coalg morph $\gamma : C \to C'$

$$f, g \in End(C', A)$$

Compare:

$$F: \quad C \xrightarrow{\gamma} C' \xrightarrow{f} A \xrightarrow{\alpha} A'$$

$$G: \quad C \xrightarrow{\gamma} C' \xrightarrow{g} A \xrightarrow{\alpha} A'$$

$$H: \quad C \xrightarrow{\gamma} C' \xrightarrow{f \star g} A \xrightarrow{\alpha} A'$$

Then

$$H = F \star G$$

pf $\forall x \in C$

write $y = \gamma(x)$

$$\Delta(y) = \sum_{(y)} y_1 \otimes y_2$$

[γ is coalg morph]

$$= \sum_{(x)} \gamma(x_1) \otimes \gamma(x_2)$$

So

$$(f \star g)(\gamma(x)) = (f \star g)(y)$$

$$= \sum_{(y)} f(y_1)\, g(y_2)$$

$$= \sum_{(x)} f(\gamma(x_1))\, g(\gamma(x_2))$$

So

$$\underbrace{\alpha\Big(\big(f \star g\big)(\gamma(x))\Big)}_{\substack{\| \\ H(x)}} = \sum_{(x)} \alpha\Big(f(\gamma(x_1))\, g(\gamma(x_2))\Big)$$

[α is alg morph]

$$= \sum_{(x)} \alpha(f(\gamma(x_1)))\, \alpha(g(\gamma(x_2)))$$

$$= \sum_{(x)} F(x_1)\, G(x_2)$$

$$= (F \star G)(x)$$

So $H = F \star G$ $\square$

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LEM IV Given

Commutative bialgebra H

Consider $id \in End(H)$.

Then for $r, s \geq 0$

$$id^{\star r} \circ id^{\star s} = id^{\star (rs)}$$

pf

id is alg morph

$id \star id$ is alg morph by LEM I

...

$id^{\star r}$ is alg morph

By LEM III (with $\alpha = id^{\star r}$, $\gamma = id$)

$$
\begin{aligned}
id^{\star r} \circ id^{\star s} &= \alpha \circ id^{\star s} \circ \gamma \\
&= (\underbrace{\alpha \circ id \circ \gamma}_{= \alpha})^{\star s} \\
&= (id^{\star r})^{\star s} \\
&= id^{\star (rs)}
\end{aligned}
$$

□