

Lec 10 Wednesday Sept 28

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LEM Assume K is a field.

Given a K -coalgebra C and a K -module U

Given a K -module hom $f: C \rightarrow U$

Consider the composition

$$\theta: C \xrightarrow{\Delta} C \otimes C \xrightarrow{id \otimes \Delta} C \otimes C \otimes C \xrightarrow{id \otimes f \otimes id} C \otimes U \otimes C$$

$$e \xrightarrow{\quad} \sum_{c_1} c_1 \otimes c_2 \xrightarrow{\quad} \sum_{c_1} c_1 \otimes c_2 \otimes c_3 \xrightarrow{\quad} \sum_{c_1} c_1 \otimes f(c_2) \otimes c_3$$

Then $\ker(\theta) \stackrel{=}{=} J$ is a subcoalgebra of C

pf Since K is field, suff to show

$$\Delta(J) \subseteq J \otimes J$$

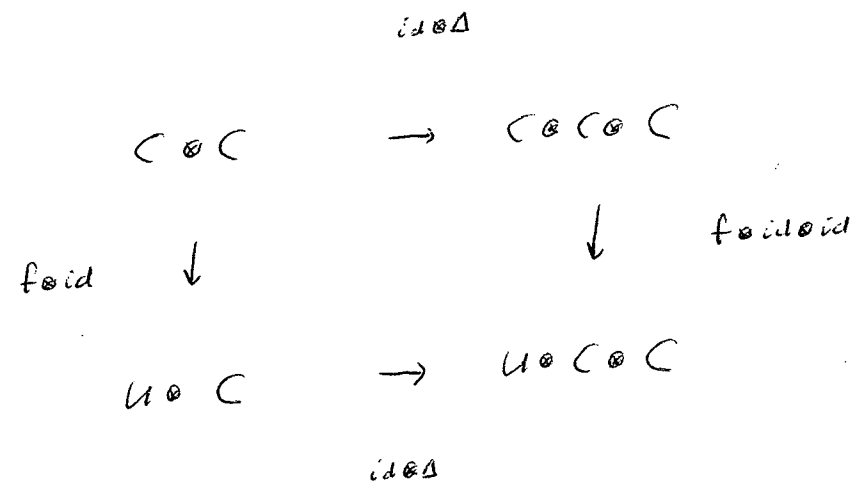
Show both

$$\Delta(J) \subseteq J \otimes C, \quad \leftarrow$$

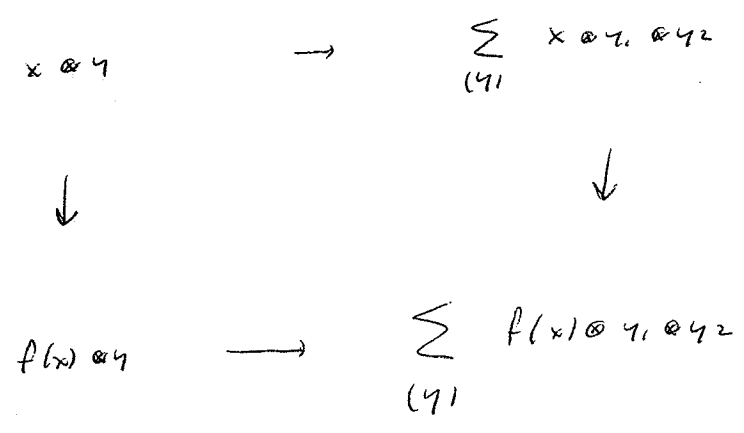
$$\Delta(J) \subseteq C \otimes J$$

The following diagrams commute:

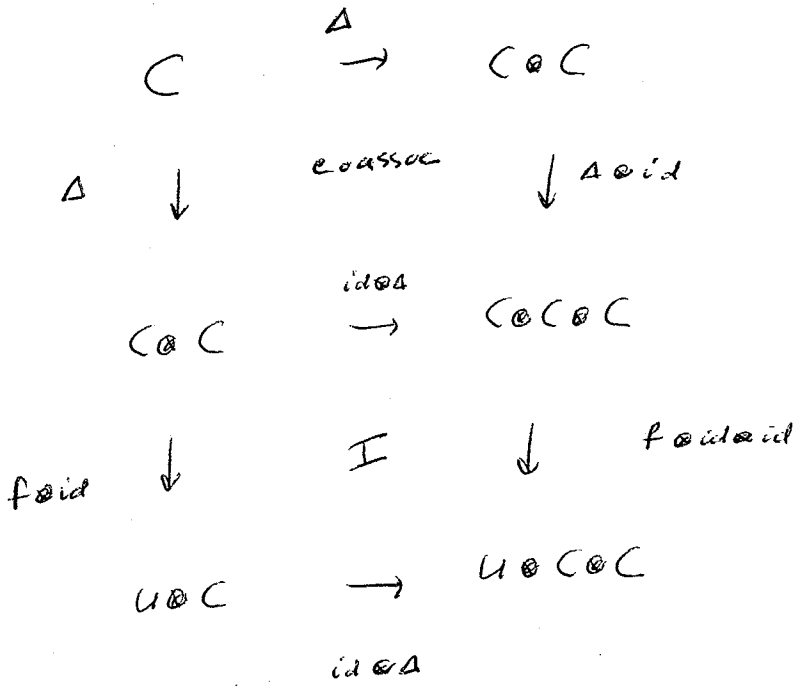
I:



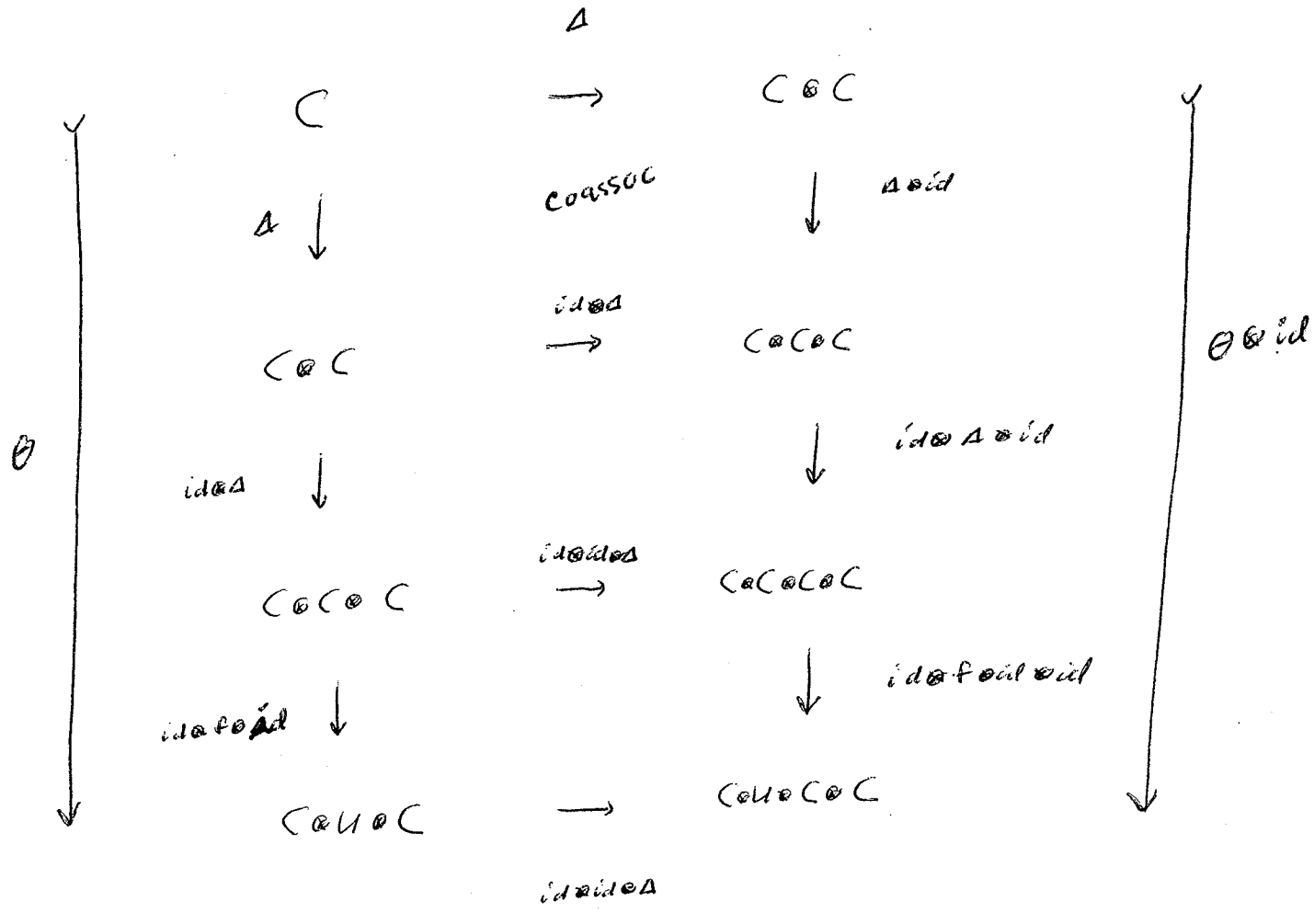
$F_n \quad x, y \in C$



II:



III ;



Bottom 2/3 is $C \otimes$ (diag II)

Chase $c \in J$ around diag III:

$$\begin{array}{ccc}
 c & \longrightarrow & \Delta(c) \\
 \downarrow & & \downarrow \\
 \downarrow & & \downarrow \\
 \downarrow & & \downarrow \\
 0 & \longrightarrow & 0
 \end{array}
 \quad \theta \otimes \text{id}$$

So $\Delta(c) \in \ker(\theta \otimes \text{id})$

$$= \underbrace{\ker(\theta)}_J \otimes C + C \otimes \underbrace{\ker(\text{id})}_0$$

$$= J \otimes C$$

So $\Delta(J) \subseteq J \otimes C$ ✓

□

LEM For above K, C, U, f, θ we have

$$\ker(\theta) \subseteq \ker(f)$$

pf For $c \in C$ st $\theta(c) = 0$
show $f(c) = 0$

$$0 = \theta(c) = \sum_{(c)} c_i \otimes f(c_i) \otimes c_3$$

Apply $\varepsilon \otimes id \otimes id$:

$$\begin{aligned} 0 &= \sum_{(c)} \varepsilon(c_i) \otimes f(c_i) \otimes c_3 \\ &= \sum_{(c)} 1 \otimes f(\varepsilon(c_i) c_i) \otimes c_3 \\ &= \sum_{(c)} 1 \otimes f(c_i) \otimes c_2 \end{aligned}$$

So $0 = \sum_{(c)} f(c_i) \otimes c_2$

Apply $id \otimes \varepsilon$:

$$\begin{aligned} 0 &= \sum_{(c)} f(c_i) \otimes \varepsilon(c_i) \\ &= f\left(\sum_{(c)} c_i \varepsilon(c_i)\right) \otimes 1 \\ &= f(c) \otimes 1 \end{aligned}$$

So $f(c) = 0$



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LEM For above K, C, U, f, θ

Given any subcoalg D of C that is contained in $\ker(f)$

then $D \subseteq \ker(\theta)$

[so $\ker(\theta)$ is the "maximal" subcoalg of C
that is contained in $\ker(f)$]

pf By assumption

$$f(D) = 0$$

For $d \in D$ show $\theta(d) = 0$

We have

$$\theta(d) = \sum_{(d)} d_1 \otimes f(d_2) \otimes d_3$$

For each summand

$$d_1, d_2, d_3 \in D.$$

In particular $d_2 \in D$ so $f(d_2) = 0$

$$\text{Now } \theta(d) = 0$$

□

LEM Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Given a non-zero ideal $I \neq H$ s.t.

$$I = \sum_{n \in \mathbb{N}} \underbrace{I \cap H_n}_{I_n}$$

then I contains a non-zero primitive element.

pf Since $I \neq 0 \exists n \in \mathbb{N}$ s.t. $I_n \neq 0$

let $N = \min \{ n \mid I_n \neq 0 \}$,

claim $N \neq 0$

pf

$\epsilon(I) = 0$ since I is coideal

$\epsilon|_{H_0} : H_0 \rightarrow K$ is bijective since H is connected

so

$0 = I \cap H_0 = I_0$

✓

let

$P =$ set of primitive elements in H

claim $I_N \subseteq P$

pf cl $I_N = I \cap H_N$ by def.

We have

$$\Delta(I) \subseteq I \otimes H + H \otimes I$$

$$= \sum_{r,2 \in \mathbb{N}} (I_r \otimes H_2 + H_2 \otimes I_r)$$

Also

$$\Delta(H_N) \subseteq \sum_{i=0}^N H_i \otimes H_{N-i}$$

So

$$\Delta(I_N) \subseteq \sum_{i=0}^N (I_i \otimes H_{N-i} + H_{N-i} \otimes I_i)$$

$$\left[I_i = 0 \text{ for } 0 \leq i \leq N-1 \right]$$

$$= I_N \otimes H_0 + H_0 \otimes I_N$$

$$\left[H_0 = k 1_H \right]$$

$$= I_N \otimes 1_H + 1_H \otimes H_N$$

For $a \in I_N$ show $a \in P$

show $\Delta(a) = a \otimes 1 + 1 \otimes a$

$$\Delta(I_N) = I_N \otimes 1 + 1 \otimes I_N$$

write

$$\Delta(a) = \psi(a) \otimes 1 + 1 \otimes \phi(a)$$

show $\phi(a) = a$

Recall

$$a = \sum_{g_i} \varepsilon(g_i) a_i$$

So

$$a = \underbrace{\sum \psi(g_i)}_0 \cdot 1 + \underbrace{\sum 1}_1 \phi(a)$$

since $\varepsilon(I_N) = 0$

So

$$a = \phi(a)$$

Sim $\psi(a) = a$

So $\Delta(a) = a \otimes 1 + 1 \otimes a$ ✓



LEM Given a connected graded bialg

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Given a graded coalg

$$C = \bigoplus_{n \in \mathbb{N}} C_n$$

Given a surj coalg morphism $f: H \rightarrow C$

show $f(H_n) \subseteq C_n$ for $n \in \mathbb{N}$ "f is graded"

TFAE

(i) f is injective

(ii) f/P is injective, where $P =$ set of prim elements in H .

pf (i) \rightarrow (ii) \checkmark

(ii) \rightarrow (i) let $I = \ker(f)$

show $I = 0$. Suppose not

We show I is coideal of H

Since f is coalg morph.

$$\begin{array}{ccc}
 H & \xrightarrow{f} & C \\
 \Delta_H \downarrow & & \downarrow \Delta_C \\
 H \otimes H & \xrightarrow{f \otimes f} & C \otimes C \\
 \uparrow & & \\
 \text{ker is } I \otimes H + H \otimes I & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 H & \xrightarrow{f} & C \\
 \varepsilon_H \downarrow & & \downarrow \varepsilon_C \\
 K & \xrightarrow{\text{id}} & K
 \end{array}$$

Require $\Delta_H(I) \subseteq I \otimes H + H \otimes I$

Require $\varepsilon_H(I) = 0$

So I is coideal of H

Since f is graded,

$$I = \sum_{n \in \mathbb{N}} I \cap H_n$$

Now by prev lemma

$$I \cap P \neq 0$$

So $f|_P$ not injective, cont.

$$\text{So } I = 0$$

□