Lec 10 Wednesday Sept 28

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LEM Assume $k$ is a field.

Given a $k$-coalg $C$ and a $k$-module $U$

Given a $k$-module hom $f : C \to U$

Consider the composition

$$\theta : \begin{array}{ccccccc} C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes C & \xrightarrow{id \otimes f \otimes id} & C \otimes U \otimes C \\ c & \longmapsto & \sum_{(c)} c_1 \otimes c_2 & \longmapsto & \sum_{(c)} c_1 \otimes c_2 \otimes c_3 & \longmapsto & \sum_{(c)} c_1 \otimes f(c_2) \otimes c_3 \end{array}$$

Then $\underset{J}{\overset{\shortparallel}{\ker(\theta)}}$ is a subcoalg of $C$

pf Since $k$ is field, suf to show

$$\Delta(J) \subseteq J \otimes J$$

Show both

$$\Delta(J) \subseteq J \otimes C, \quad \leftarrow$$

$$\Delta(J) \subseteq C \otimes J$$

The following diagrams commute:

I :

$$\begin{tikzcd} C \otimes C \arrow[r, "id \otimes \Delta"] \arrow[d, "f \otimes id"'] & C \otimes C \otimes C \arrow[d, "f \otimes id \otimes id"] \\ U \otimes C \arrow[r, "id \otimes \Delta"'] & U \otimes C \otimes C \end{tikzcd}$$

Fn $x, y \in C$

$$\begin{tikzcd} x \otimes y \arrow[r] \arrow[d] & \sum_{(y)} x \otimes y_1 \otimes y_2 \arrow[d] \\ f(x) \otimes y \arrow[r] & \sum_{(y)} f(x) \otimes y_1 \otimes y_2 \end{tikzcd}$$

✓

II:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\text{coassoc}"] & C\otimes C \arrow[d, "\Delta\otimes id"] \\
C\otimes C \arrow[r, "id\otimes\Delta"] \arrow[d, "f\otimes id"'] \arrow[dr, phantom, "\mathrm{I}"] & C\otimes C\otimes C \arrow[d, "f\otimes id\otimes id"] \\
U\otimes C \arrow[r, "id\otimes\Delta"'] & U\otimes C\otimes C
\end{tikzcd}$$

## III :

$$
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\text{coassoc}"] \arrow[ddd, bend right=60, "\theta"'] & C\otimes C \arrow[d, "\Delta\otimes id"] \arrow[ddd, bend left=60, "\theta\otimes id"] \\
C\otimes C \arrow[r, "id\otimes\Delta"] \arrow[d, "id\otimes\Delta"'] & C\otimes C\otimes C \arrow[d, "id\otimes\Delta\otimes id"] \\
C\otimes C\otimes C \arrow[r, "id\otimes id\otimes\Delta"] \arrow[d, "id\otimes f\otimes id"'] & C\otimes C\otimes C\otimes C \arrow[d, "id\otimes f\otimes id\otimes id"] \\
C\otimes U\otimes C \arrow[r, "id\otimes id\otimes\Delta"'] & C\otimes U\otimes C\otimes C
\end{tikzcd}
$$

Bottom 2/3 is $C\otimes(\text{diag II})$

Chase $c \in J$ around diag III:

$$\begin{tikzcd}
c \arrow[r] \arrow[d, "\theta"'] & \Delta(c) \arrow[d, "\theta \otimes id"] \\
0 \arrow[r] & 0
\end{tikzcd}$$

So $\Delta(c) \in \ker(\theta \otimes id)$

$$= \underset{J}{\underset{\shortparallel}{\ker(\theta)}} \otimes C + C \otimes \underset{0}{\underset{\shortparallel}{\ker(id)}}$$

$$= J \otimes C$$

So $\Delta(J) \subseteq J \otimes C$ ✓

$\square$

LEM For above $K, C, U, f, \theta$ we have

$$\ker(\theta) \subseteq \ker(f)$$

pf Fix $c \in C$ st $\theta(c) = 0$

show $f(c) = 0$

$$0 = \theta(c) = \sum_{(c)} c_1 \otimes f(c_2) \otimes c_3$$

Apply $\varepsilon \otimes id \otimes id$:

$$0 = \sum_{(c)} \varepsilon(c_1) \otimes f(c_2) \otimes c_3$$

$$= \sum_{(c)} 1 \otimes f(\varepsilon(c_1) c_2) \otimes c_3$$

$$= \sum_{(c)} 1 \otimes f(c_1) \otimes c_2$$

So $$0 = \sum_{(c)} f(c_1) \otimes c_2$$

Apply $id \otimes \varepsilon$:

$$0 = \sum_{(c)} f(c_1) \otimes \varepsilon(c_2)$$

$$= f\left(\sum_{(c)} c_1 \varepsilon(c_2)\right) \otimes 1$$

$$= f(c) \otimes 1$$

So $f(c) = 0$

$\square$

LEM For above $k, C, U, f, \theta$

Given any subcoalg $D$ of $C$ that is contained in $\ker(f)$

then $D \subseteq \ker(\theta)$

[ so $\ker(\theta)$ is the "maximal" subcoalg of $C$ that is contained in $\ker(f)$ ]

pf By assumption

$$f(D) = 0$$

For $d \in D$ show $\theta(d) = 0$

We have

$$\theta(d) = \sum_{(d)} d_1 \otimes f(d_2) \otimes d_3$$

For each summand

$$d_1, d_2, d_3 \in D.$$

In particular $d_2 \in D$ so $f(d_2) = 0$

Now $\theta(d) = 0$ $\square$

LEM Given a connected graded bialgebra

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Given a nonO coideal $I$ of $H$ st

$$I = \sum_{n \in \mathbb{N}} \underbrace{I \cap H_n}_{\substack{\shortparallel \\ I_n}}$$

Then $I$ contains a nonO primitive element.

pf Since $I \neq 0$ $\exists$ $n \in \mathbb{N}$ st $I_n \neq 0$

let $N = \min\{n \mid I_n \neq 0\}$.

claim $N \neq 0$

pf cl $\varepsilon(I) = 0$ since $I$ is coideal

$\varepsilon|_{H_0} : H_0 \to K$ is bij since $H$ is connected

So

$$0 = I \cap H_0 = I_0$$

✓

Let $P$ = set of primitive elements in $H$

Claim $I_N \subseteq P$

pf cl $\quad I_N = I \cap H_N \quad$ by def.

We have
$$\Delta(I) \subseteq I \otimes H + H \otimes I$$
$$= \sum_{r,s \in \mathbb{N}} \left( I_r \otimes H_s + H_s \otimes I_r \right)$$

Also
$$\Delta(H_N) \subseteq \sum_{i=0}^{N} H_i \otimes H_{N-i}$$

So
$$\Delta(I_N) \subseteq \sum_{i=0}^{N} \left( I_i \otimes H_{N-i} + H_{N-i} \otimes I_i \right)$$
$$[\, I_i = 0 \text{ for } 0 \leq i \leq N-1 \,]$$
$$= I_N \otimes H_0 + H_0 \otimes I_N \qquad [\, H_0 = k 1_H \,]$$
$$= I_N \otimes 1_H + 1_H \otimes H_N$$

For $a \in I_N$ show $a \in P$

show $\Delta(a) = a \otimes 1 + 1 \otimes a$

$$\Delta(I_N) \subseteq I_N \otimes 1 + 1 \otimes I_N$$

write $\Delta(a) = \varphi(a) \otimes 1 + 1 \otimes \phi(a)$

show $\phi(a) = a$

Recall

$$a = \sum_{(a)} \varepsilon(a_1)\, a_2$$

So

$$a = \underbrace{\varepsilon(\varphi(a))}_{\substack{\shortparallel \\ 0}} 1 + \underbrace{\varepsilon(1)}_{\substack{\shortparallel \\ 1}} \phi(a)$$

since $\varepsilon(I_N) = 0$

So $a = \phi(a)$

Sim $\varphi(a) = a$

So $\Delta(a) = a \otimes 1 + 1 \otimes a$ ✓

$\square$

LEM Given a connected graded bialg

$$H = \bigoplus_{n \in \mathbb{N}} H_n$$

Given a graded coalg

$$C = \bigoplus_{n \in \mathbb{N}} C_n$$

Given a surj coalg morphism $f : H \to C$

s.t. $f(H_n) \subseteq C_n$ for $n \in \mathbb{N}$ "$f$ is graded"

TFAE

(i) $f$ is injective

(ii) $f|_P$ is injective, where $P$ = set of prim elements in $H$.

pf (i) → (ii) ✓

(ii) → (i) let $I = \ker(f)$

show $I = 0$. Suppose not.

We show $I$ is coideal of $H$

Since $f$ is coalg morph,

$$\begin{array}{ccc} H & \xrightarrow{f} & C \\ \Delta_H \downarrow & & \downarrow \Delta_C \\ H\otimes H & \xrightarrow[f\otimes f]{} & C\otimes C \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{f} & C \\ \varepsilon_H \downarrow & & \downarrow \varepsilon_C \\ K & \xrightarrow[id]{} & K \end{array}$$

ker is $I\otimes H + H\otimes I$

Require $\Delta_H(I) \subseteq I\otimes H + H\otimes I$

Require $\varepsilon_H(I) = 0$

So $I$ is coideal of $H$

Since $f$ is graded,

$$I = \sum_{n\in\mathbb{N}} I\cap H_n$$

Now by prev lemma

$$I\cap P \neq 0$$

So $f|_P$ not injective, cont,

So $I = 0$

$\square$