

Lec 1

Wednesday Sept 7

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Fall 2016

Math 846: Hopf Algebras in Combinatorics

MWF 11:00 AM in B235 VV

Paul Terwilliger

Office hrs: By apt or find me in
101A, 322

Text: Grinberg + Reiner, Hopf Algebras in Combin

We will go through the notes

Grading: Near end of semester, each nondissertator
will give the lecture on a topic of their choice
or from the text (details later)

We assume you are familiar with

groups, rings, modules, tensor products, ...

For review see

Dummit + Foote, Abstract Algebra

Throughout,

K is a commutative ring with 1

$$\otimes = \otimes_K$$

Recall for a K -module V

• $V, +, 0$ is an abelian group

• The action $K \times V \rightarrow V$ satisfies

$$\alpha v \rightarrow \alpha v$$

$$\alpha(u+v) = \alpha u + \alpha v$$

$$(\alpha + \beta)v = \alpha v + \beta v$$

$$(\alpha\beta)v = \alpha(\beta v)$$

$$1v = v$$

$$\alpha, \beta \in K$$

$$u, v \in V$$

Ex $V=K$ is a K -module with action

$$K \times K \rightarrow K$$

$$\alpha v \rightarrow \alpha v$$

REV Given K -modules U, V

their direct sum

$$U \oplus V = \left\{ (u, v) \mid u \in U, v \in V \right\}$$

is a K -module with action

$$\alpha(u, v) = (\alpha u, \alpha v)$$

Also their tensor product $U \otimes V$

is a K -module with action

$$\alpha(u \otimes v) = (\alpha u) \otimes v = u \otimes (\alpha v)$$

Given K -modules V, V'

A map $\sigma: V \rightarrow V'$ is a K -module homomorphism

whenever

$$\sigma(u+v) = \sigma(u) + \sigma(v)$$

$$u, v \in V$$

$$\sigma(\alpha u) = \alpha \sigma(u)$$

$$\alpha \in K$$

A K -module isomorphism is a bijective K -module hom.

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Given K -modules U, V

The map

$$\begin{aligned} \tau_{U,V} : U \otimes V &\rightarrow V \otimes U \\ u \otimes v &\rightarrow v \otimes u \end{aligned}$$

is a K -module iso.

K-algebras (version I)

K-algebra A satisfies

• A is K-module

• the mult $A \times A \rightarrow A$
 $a \ b \rightarrow ab$

is assoc and K-linear in each argument:

$$a(bc) = (ab)c$$

$a, b, c \in A$

$$a(b+c) = ab+ac$$

$\lambda \in K$

$$(a+b)c = ac+bc$$

$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$

• $\exists 0 \neq 1_A \in A$ such that

$$a 1_A = a = 1_A a \quad \forall a \in A$$

For a k -algebra A observe:

• The map

$$\begin{array}{lcl}
 u: & k & \rightarrow A & \text{"unit map"} \\
 & \alpha & \rightarrow \alpha 1_A &
 \end{array}$$

is a k -module hom.

• The multiplication induces a k -module hom

$$\begin{array}{lcl}
 m: & A \otimes A & \rightarrow A & \text{"mult map"} \\
 & a \otimes b & \rightarrow ab & \text{"product map"}
 \end{array}$$

We now describe k -algebras using u, m

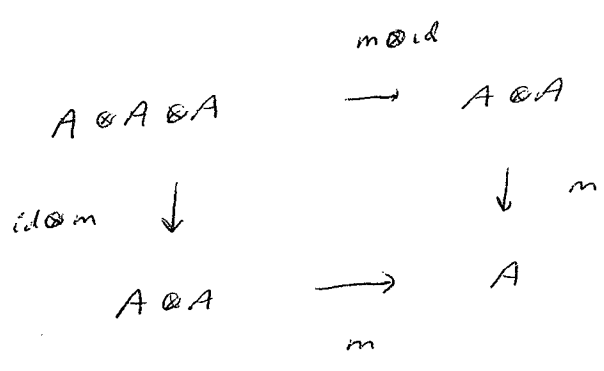
K-algebras (version II)

K-algebra A is a non 0 K-module

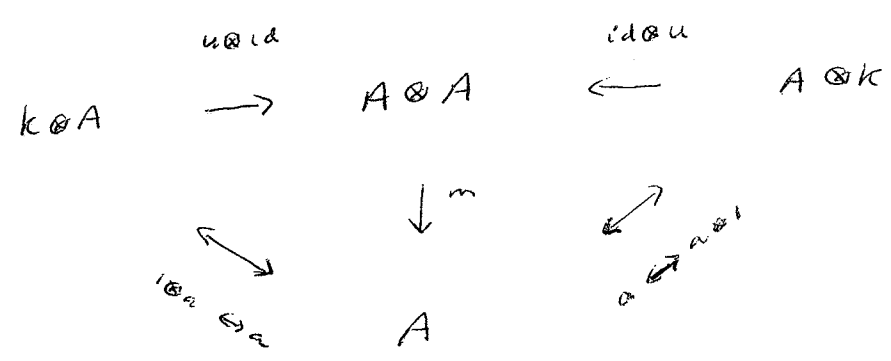
together with K-module homs

$$u: K \rightarrow A, \quad m: A \otimes A \rightarrow A$$

that make these diagrams commute:



(*)



(**)

One checks versions I, II are equiv.

Examples of K -algebras

Given a K -module V

For $n \in \mathbb{N}$

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

define

$$V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n \text{ copies}}$$

K -module

view

$$V^{\otimes 0} = K$$

define

$$T(V) = \bigoplus_{n \in \mathbb{N}} T^{\otimes n}$$

K -module

$T(V)$ becomes a K -algebra with product

$$V^{\otimes r} \times V^{\otimes s} \rightarrow V^{\otimes (r+s)}$$

$$R \quad S \quad \rightarrow \quad R \otimes S$$

and ident

$$1 \in K = V^{\otimes 0} \subseteq T(V)$$

Call $T(V)$ the tensor algebra on V

Identify

$$V = V^{\otimes 1} \subseteq T(V)$$

Notation for $T(V)$

Given $v_1, v_2, \dots, v_n \in V$

Consider

$$v_1 \otimes v_2 \otimes \dots \otimes v_n \in V^{\otimes n} \subseteq T(V)$$

||

$$v_1 v_2 \dots v_n \quad (\text{product in alg } T(V))$$

↗

often use this notation

REV

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Given a K -alg A

Recall a 2-sided ideal of A is a

K -submodule $J \subseteq A$ st both

$$AJ \subseteq J,$$

$$JA \subseteq J.$$

Given a 2-sided ideal $J \subseteq A$ with $J \neq A$

the quotient A/J is K -alg with mult

$$A/J \times A/J \rightarrow A/J$$

$$a+J \quad b+J \rightarrow ab+J$$

and ident

$$1_A + J$$

Given K -module V

let

$J = 2$ -sided ideal of $T(V)$ gen by

$$uv - vu \quad u, v \in V$$

Obs

$$J \subseteq \bigoplus_{n \geq 2} V^{\otimes n}$$

So

$$J \neq T(V)$$

the quot algebra

$$S(V) = T(V)/J$$

"

$$S(V)$$

is called the symmetric algebra on V

Obs $S(V)$ is commutative

the canonical map

$$\begin{aligned} \text{can: } T(V) &\rightarrow S(V) \\ x &\rightarrow x + J \end{aligned}$$

is surj K -alg morphism with kernel J

Ideal J is gen by elements in $V^{\otimes 2}$

So $J = \sum_{n \in \mathbb{N}} (J \cap V^{\otimes n})$ ds *

and

$J \cap V^{\otimes 0} = 0, \quad J \cap V^{\otimes 1} = 0$

For $n \in \mathbb{N}$ define

$Sym^n(V) = \text{image of } V^{\otimes n} \text{ under } \text{can}$

The restr

$\text{can} / V^{\otimes n} : V^{\otimes n} \rightarrow Sym^n(V)$ **

is surj k module hom with ker $J \cap V^{\otimes n}$

So ** is bijectum for $n=0,1$

Via: ** identify

$k \equiv Sym^0(V), \quad V \equiv Sym^1(V)$

By * the sum

$$\text{Sym}(V) = \sum_{n \in \mathbb{N}} \text{Sym}^n(V)$$

is direct,

By constr.

$$\text{Sym}^r(V) \text{Sym}^s(V) \subseteq \text{Sym}^{r+s}(V) \quad r, s \in \mathbb{N}$$

Given a group G

Let KG denote the set of formal sums

$$\sum_{g \in G} \alpha_g t_g \quad \alpha_g \in K \quad t_g \text{ symbol}$$

(finitely many α_g non 0)

KG is K -alg with mult

$$t_g t_h = t_{gh} \quad g, h \in G$$

and ident

t_e

$e = \text{ident of } G$

KG is the group algebra of G over K

REV Given K -algebras A, B

then $A \otimes B$ is a K -alg with product

$$A \otimes B \times A \otimes B \rightarrow A \otimes B$$

$$a \otimes b \quad a' \otimes b' \rightarrow (aa') \otimes (bb')$$

and ident

$$1_{A \otimes B} = 1_A \otimes 1_B$$

DEF A k-coalgebra is a

non-zero k-module C together with k-module maps

$$\Delta: C \rightarrow C \otimes C$$

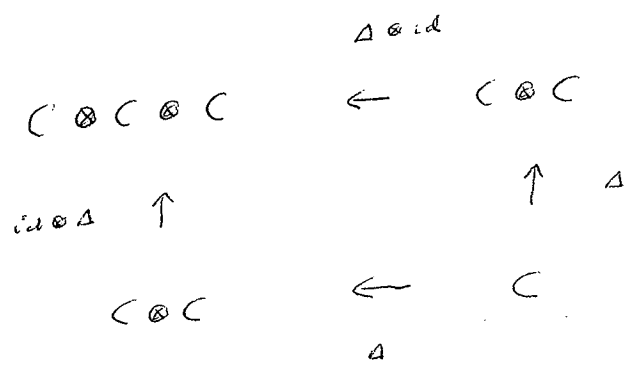
"comult"

"co-product"

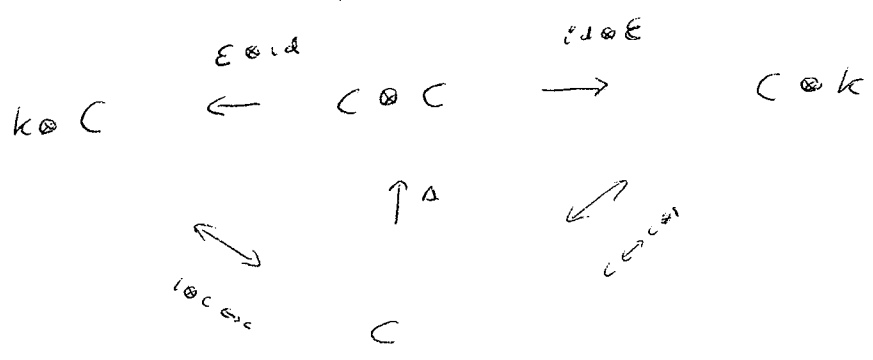
$$\epsilon: C \rightarrow k$$

"counit"

such that these diagrams commute:



(*)
"coassoc"



(**)

"counit"

For $c \in C$ write

$$\Delta(c) = \sum_{(c_1)} c_1 \otimes c_2$$

" Sweedler notation "

(**) becomes

$$\sum_{(c_1)} \varepsilon(c_1) c_2 = c = \sum_{(c_1)} c_1 \varepsilon(c_2)$$

In (*) chase c around diagram.

For either path, write final image as

$$\sum_{(c)} c_1 \otimes c_2 \otimes c_3$$

So

$$\begin{aligned} \sum_{(c)} c_1 \otimes c_2 \otimes c_3 &= \sum_{(c_1)} \Delta(c_1) \otimes c_2 \\ &= \sum_{(c_1)} c_1 \otimes \Delta(c_2) \end{aligned}$$

Similarly write

$$\begin{aligned} \sum_{(c)} c_1 \otimes c_2 \otimes c_3 \otimes c_4 &= \sum_{(c_1)} \Delta(c_1) \otimes c_2 \otimes c_3 \\ &= \sum_{(c_1)} c_1 \otimes \Delta(c_2) \otimes c_3 \\ &= \sum_{(c_1)} c_1 \otimes c_2 \otimes \Delta(c_3) \end{aligned}$$

etc.