Fall 2016

Math 846: Hopf Algebras in Combinatorics

MWF 11:00 AM in B235 VV

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Office hrs: By apt or find me in
101A, 322

Text: Grinberg + Reiner. Hopf Algebras in Combin

We will go through the notes

Grading: Near end of semester, each nondissertator
will give the lecture on a topic of their choice
or from the text (details later)

We assume you are familiar with
groups, rings, modules, tensor products, ...

For review see
Dummit + Foote. Abstract Algebra

Throughout,

K is a commutative ring with 1

$\otimes = \otimes_K$

<u>Recall</u> for a <u>K-module V</u>

- $V, +, 0$ is an abelian group
- the action $\begin{matrix} K \times V \to V \\ \alpha \quad v \to \alpha v \end{matrix}$ satisfies

$$\alpha(u+v) = \alpha u + \alpha v$$

$$(\alpha+\beta)v = \alpha v + \beta v \qquad \alpha, \beta \in K \quad u, v \in V$$

$$(\alpha\beta)v = \alpha(\beta v)$$

$$1 v = v$$

Ex $V = K$ is a K-module with action

$$\begin{matrix} K \times K \to K \\ \alpha \quad v \to \alpha v \end{matrix}$$

REV Given k-modules U, V

their direct sum

$$U \oplus V = \{ (u,v) \mid u \in U, \ v \in V \}$$

is a k-module with action

$$\alpha (u,v) = (\alpha u, \alpha v)$$

Also their tensor product $U \otimes V$

is a k-module with action

$$\alpha (u \otimes v) = (\alpha u) \otimes v = u \otimes (\alpha v)$$

Given k-modules V, V'

A map $\sigma : V \to V'$ is a <u>k-module homomorphism</u>

whenever

$$\sigma(u+v) = \sigma(u) + \sigma(v) \qquad u, v \in V$$

$$\sigma(\alpha u) = \alpha \sigma(u) \qquad \alpha \in k$$

A <u>k-module isomorphism</u> is a bijective k-module hom.

Given k-modules U, V

the map

$$\begin{aligned} U \otimes V &\to V \otimes U \\ T_{U,V} : \quad u \otimes v &\mapsto v \otimes u \end{aligned}$$

is a k-module iso.

K-algebras (version I)

k-algebra A satisfies

- A is K-module
- the mult $\begin{array}{ccc} A \times A & \to & A \\ a \quad b & \to & ab \end{array}$

  is assoc and k-linear in each argument:

  $a(bc) = (ab)c$

  $a(b+c) = ab + ac$ $\qquad a, b, c \in A$

  $(a+b)c = ac + bc$ $\qquad \alpha \in K$

  $(\alpha a)b = \alpha(ab) = a(\alpha b)$

- $\exists \; 0 \neq 1_A \in A$ such that

  $a \, 1_A = a = 1_A a \qquad \forall a \in A$

For a k-algebra A observe:

- The map

$$u: \begin{array}{ccc} k & \to & A \\ \alpha & \mapsto & \alpha 1_A \end{array}$$
"unit map"

is a k-module hom.

- The multiplication induces a k-module hom

$$m: \begin{array}{ccc} A \otimes A & \to & A \\ a \otimes b & \to & ab \end{array}$$
"mult map"

"product map"

We now describe k-algebras using u, m

<u>k-algebras</u> (version II)

k-algebra A is a nmo k-module

together with k-module homs

$$u : k \to A, \qquad m : A \otimes A \to A$$

that make these diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd} \qquad (*)$$

$$\begin{tikzcd}
k \otimes A \arrow[r, "u \otimes id"] \arrow[dr, leftrightarrow, "1 \otimes a \leftrightarrow a"'] & A \otimes A \arrow[d, "m"] & A \otimes k \arrow[l, "id \otimes u"'] \arrow[dl, leftrightarrow, "a \leftrightarrow a \otimes 1"] \\
& A &
\end{tikzcd} \qquad (**)$$

One checks versions I, II are equiv.

Examples of k-algebras

Given a k-module V

For $n \in \mathbb{N}$ $\qquad \mathbb{N} = \{0, 1, 2, \dots\}$

define

$$V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ copies}} \qquad \text{k-module}$$

View

$$V^{\otimes 0} = k$$

Define

$$T(V) = \bigoplus_{n \in \mathbb{N}} T^{\otimes n} \qquad \text{k-module}$$

$T(V)$ beccomes a k-algebra with product

$$\begin{array}{ccccc} V^{\otimes r} & \times & V^{\otimes s} & \longrightarrow & V^{\otimes (r+s)} \\ R & & S & \to & R \otimes S \end{array}$$

and ident

$$1 \in k = V^{\otimes 0} \subseteq T(V)$$

Call $T(V)$ the <u>tensor algebra on V</u>

Identify

$$V = V^{\otimes 1} \subseteq T(V)$$

Notation for $T(V)$

Given $v_1, v_2, \ldots, v_n \in V$

Consider

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n} \subseteq T(V)$$

$$\parallel$$

$$v_1 v_2 \cdots v_n \quad (\text{product in alg } T(V))$$

often use this notation

REV

Given a k-alg A

Recall a <u>2-sided ideal of A</u> is a

K-submodule $J \subseteq A$ st both

$$AJ \subseteq J,$$

$$JA \subseteq J.$$

Given a 2-sided ideal $J \subseteq A$ with $J \neq A$

the quotient A/J is k-alg with mult

$$\begin{array}{ccccc} A/J & \times & A/J & \to & A/J \\ a+J & & b+J & \to & ab+J \end{array}$$

and ident

$$1_A + J$$

Given K-module V

let
$$J = \text{2-sided ideal of } T(V) \text{ gen by}$$
$$uv - vu \qquad u, v \in V$$

Obs
$$J \subseteq \bigoplus_{n \geq 2} V^{\otimes n}$$

So
$$J \neq T(V)$$

The quot algebra
$$S(V) = T(V)/J$$
$$\parallel$$
$$\text{Sym}(V)$$

is called the <u>symmetric algebra</u> on V

Obs $S(V)$ is commutative

The canonical map
$$\text{Can}: \begin{array}{ccc} T(V) & \to & S(V) \\ x & \to & x + J \end{array}$$

is surj K-alg morphism with kernel J

Ideal J is gen by elements in $V^{\otimes 2}$

so $$J = \sum_{n \in \mathbb{N}} (J \cap V^{\otimes n}) \quad \text{ds}$$ $*$

and $$J \cap V^{\otimes 0} = 0, \qquad J \cap V^{\otimes 1} = 0$$

Fn $n \in \mathbb{N}$ define

$$\mathrm{Sym}^n(V) = \text{image of } V^{\otimes n} \text{ under can}$$

the restr

$$\mathrm{can}|_{V^{\otimes n}} : V^{\otimes n} \longrightarrow \mathrm{Sym}^n(V) \qquad **$$

is surj k module hom with ker $J \cap V^{\otimes n}$

So $**$ is bijection fn $n = 0, 1$

Via $**$ identify

$$k \equiv \mathrm{Sym}^0(V), \qquad V \equiv \mathrm{Sym}^1(V)$$

By * the sum

$$\mathrm{Sym}(V) = \sum_{n \in \mathbb{N}} \mathrm{Sym}^n(V)$$

is direct.

By cnstr

$$\mathrm{Sym}^r(V)\,\mathrm{Sym}^s(V) \subseteq \mathrm{Sym}^{r+s}(V) \qquad r,s \in \mathbb{N}$$

Given a group G

Let kG denote the set of formal sums

$$\sum_{g \in G} \alpha_g t_g \qquad \alpha_g \in k \qquad t_g \text{ symbol}$$

(finitely many α_g nonzero)

kG is k-alg with mult

$$t_g t_h = t_{gh} \qquad g, h \in G$$

and ident

$$t_e \qquad e = \text{ident of } G$$

kG is the <u>group algebra of G over k</u>

REV Given k-algebras A, B

then $A \otimes B$ is a k-alg with product

$$A \otimes B \times A \otimes B \longrightarrow A \otimes B$$

$$a \otimes b \quad a' \otimes b' \longrightarrow (aa') \otimes (bb')$$

and ident

$$1_{A \otimes B} = 1_A \otimes 1_B$$

DEF A <u>k-coalgebra</u> is a nonO k-module C together with k-module homs

$$\Delta : C \longrightarrow C \otimes C$$

"comult"
"coproduct"

$$\varepsilon : C \longrightarrow k$$

"counit"

such that these diagrams commute:

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes id"'] \\
C \otimes C \arrow[u, "id \otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

(*)

"coassoc"

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes k \\
& C \arrow[u, "\Delta"] \arrow[ul, leftrightarrow, "1 \otimes c \leftrightarrow c"] \arrow[ur, leftrightarrow, "c \leftrightarrow c \otimes 1"'] &
\end{tikzcd}$$

(**)

"counit"

For $c \in C$ write

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2$$ "Sweedler notation"

$(**)$ becomes

$$\sum_{(c)} \varepsilon(c_1) c_2 = c = \sum_{(c)} c_1 \varepsilon(c_2)$$

In $(*)$ chase c around diagram.

For either path, write final image as

$$\sum_{(c)} c_1 \otimes c_2 \otimes c_3$$

So

$$\sum_{(c)} c_1 \otimes c_2 \otimes c_3 = \sum_{(c)} \Delta(c_1) \otimes c_2$$

$$= \sum_{(c)} c_1 \otimes \Delta(c_2)$$

Similarly write

$$\sum_{(c)} c_1 \otimes c_2 \otimes c_3 \otimes c_4 = \sum_{(c)} \Delta(c_1) \otimes c_2 \otimes c_3$$

$$= \sum_{(c)} c_1 \otimes \Delta(c_2) \otimes c_3$$

$$= \sum_{(c)} c_1 \otimes c_2 \otimes \Delta(c_3)$$

etc.